

Finite-size corrections to the rotating string and the winding state

Davide Astolfi,^a Troels Harmark,^b Gianluca Grignani^a and Marta Orselli^b

^a*Dipartimento di Fisica, Università di Perugia and
I.N.F.N. Sezione di Perugia,
Via Pascoli, I-06123 Perugia, Italy*

^b*The Niels Bohr Institute,
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
E-mail: astolfi@pg.infn.it, harmark@nbi.dk, grignani@pg.infn.it,
orselli@nbi.dk*

ABSTRACT: We compute higher order finite size corrections to the energies of the circular rotating string on $AdS_5 \times S^5$, of its orbifolded generalization on $AdS_5 \times S^5/\mathbb{Z}_M$ and of the winding state which is obtained as the limit of the orbifolded circular string solution when $J \rightarrow \infty$ and J/M^2 is kept fixed. We solve, at the first order in $\lambda' = \lambda/J^2$, where λ is the 't Hooft coupling, the Bethe equations that describe the anomalous dimensions of the corresponding gauge dual operators in an expansion in m/K , where m is the winding number and K is the “magnon number”, and to all orders in the angular momentum J . The solution for the circular rotating string and for the winding state can be matched to the energy computed from an effective quantum Landau-Lifshitz model beyond the first order correction in $1/J$. For the leading $1/J$ corrections to the circular rotating string in m^2 and m^4 and for the subleading $1/J^2$ corrections to the m^2 term, we find agreement. For the winding state we match the energy completely up to, and including, the order $1/J^2$ finite-size corrections.

The solution of the Bethe equations corresponding to the spinning closed string is also provided in an expansion in m/K and to all orders in J .

KEYWORDS: Penrose limit and pp-wave background, AdS-CFT Correspondence.

Contents

1.	Introduction	1
2.	Finite-size corrections to the circular/spinning string state	5
2.1	All-order finite-size effects from Bethe equations	6
2.2	Results from Landau-Lifshitz sigma-model	10
3.	Finite-size corrections to the winding state	15
3.1	Winding state from Bethe equations	15
3.2	Winding state from Landau-Lifshitz sigma-model	19
4.	Conclusions	25
A.	Zeroes of the Laguerre polynomial	27
B.	Circular rotating string	27
C.	Parameterizations of the two-sphere	29

1. Introduction

Semi-classical closed string states on $AdS_5 \times S^5$ [1–3] and their gauge theory duals, local composite operators of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, have recently played an important role in the understanding of the AdS/CFT correspondence. The discovery of integrable structures in planar $\mathcal{N} = 4$ SYM theory [4–7] and tree-level string theory on $AdS_5 \times S^5$ [8, 9] has sparked the hope of being able to match the spectrum of semi-classical string states with that of their dual gauge theory operators.

Considering as an example semi-classical string states with a large angular momentum J on S^5 , corresponding to an R-charge in $\mathcal{N} = 4$ SYM, one can have $\lambda' = \lambda/J^2$ small (λ being the 't Hooft coupling of $\mathcal{N} = 4$ SYM theory) on both sides of the correspondence, in gauge theory by expanding in λ , and on the string side by expanding in $1/J$ in the semi-classical regime $\lambda \gg 1$. While agreement is found at first and second order in λ' , for the leading and the first $1/J$ correction, the agreement breaks down for λ'^3 , a disagreement known as the three-loop discrepancy [10].¹ Recently a substantial effort has been made to remedy this disagreement, in order to establish an interpolation between weak and strong 't Hooft coupling, by the introduction of the so-called dressing phase factor [12–14].

¹See also [11] for a closely related discrepancies in the near plane wave/BMN correspondence also cured by the introduction of the dressing factor.

Another question, that has received somewhat less attention, is to what extent gauge theory and string theory agree to first order in λ' . As stated above, agreement has been found up to first order in $1/J$ [11].² However, the agreement has not been tested beyond first order in $1/J$. It has been conjectured in [12] that the planar gauge theory and tree-level string theory agree exactly to all orders in $1/J$ in the sense that the same Bethe equations and dispersion relation describe both. From the string theory point of view this is interesting since one should see the emergence of the discrete nature of the string worldsheet from an $E = p^2$ to an $E = 4 \sin^2(p/2)$ type of dispersion relation (in the $\mathfrak{su}(2)$ sector).

In this paper we explore this question for the case of the semi-classical circular closed string state [3, 21, 22] and furthermore for its orbifolded generalization. The circular string that we consider is confined in a $\mathbb{R} \times S^3$ subspace of $\text{AdS}_5 \times S^5$, with the S^3 being inside S^5 . The circular string in this subspace has two independent angular momenta J_1 and J_2 corresponding to the two rotation angles ϕ_1 and ϕ_2 of the S^3 . On the gauge theory side, these angular momenta are identified with two of the R -charges. The string has large $J = J_1 + J_2$ but with the ratio $\alpha = J_2/J$ fixed. The circular string is characterized by having a non-zero winding number m with respect to the angle $\varphi = \phi_1 - \phi_2$.

On the gauge theory side the circular string state is mapped to an operator in the $\mathfrak{su}(2)$ sector, being $\text{Tr}(Z^{J_1} X^{J_2})$ or permutations thereof, where Z and X are two of the complex scalars of $\mathcal{N} = 4$ SYM theory. The one-loop scaling dimensions of operators in the $\mathfrak{su}(2)$ sector are described exactly by the $XXX_{1/2}$ ferromagnetic Heisenberg spin chain and its corresponding Bethe equations.

One of the aims of this paper is to match higher order corrections in $1/J$ between the circular string state and the corresponding gauge theory operator, for the part of the energy which is first-order in λ' . However, it is difficult to understand such corrections in the full quantum string theory, since that requires to include modes outside the $\mathbb{R} \times S^3$ subspace. Instead, we adopt in this paper the approach of [23, 24] and use the Landau-Lifshitz sigma-model [25–27], plus certain higher derivative terms, as an effective description of the string side. This is furthermore known to be a long wave-length approximation to the $XXX_{1/2}$ ferromagnetic Heisenberg spin chain which we have on the gauge theory side.

We consider first the circular string using the Bethe equations for the $XXX_{1/2}$ ferromagnetic Heisenberg spin chain. This has previously been considered in [28, 29]. Since we are interested in finding higher order corrections in $1/J$, we employ a novel way of solving the Bethe equations. This consists of making an expansion in powers of the winding number m while at the same time having J large. We obtain in this way the m^2 and m^4 contributions to the energy

$$\frac{E - J}{\lambda'} = \frac{1 - \alpha}{\alpha} \frac{Jm^2}{2(1 - 1/J)} \left(1 - \frac{(1 - \alpha)}{\alpha} \frac{\pi^2 m^2}{3(J - 1)} \right) + \mathcal{O}(m^6) \quad (1.1)$$

This result is consistent with previous results for circular strings [28, 29]. Notice that eq. (1.1) contains the corrections in $1/J$ to any order. The novel procedure that we use to solve the Bethe equations takes advantage of some exact properties of the zeroes of the

²In [15] an argument is given for why, at the order λ' , gauge theory and string theory agree up to first order in $1/J$. This argument is based the so-called decoupling limit [16–20] of AdS/CFT.

Laguerre polynomial [30] and, being quite powerful, it could presumably be extended also to higher powers of the winding number of the string states. However, it is important to remark that for finite J one needs to add additional contributions that are non-perturbative in $1/J$. These non-perturbative contributions are related to the instabilities of the circular string [28, 29, 23].

Since the solution of the Bethe equations for the $XXX_{1/2}$ Heisenberg spin chain, describing the one-loop contribution to the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM theory, is closely related to the Bethe equations for the $XXX_{-1/2}$ Heisenberg spin chain, which instead describes the one-loop contribution to the $\mathfrak{sl}(2)$ sector, we solve both sectors at the same time. The $\mathfrak{sl}(2)$ sector consists of operators of the type $\text{Tr}(D^{s_1} Z D^{s_2} Z \dots D^{s_J} Z)$. Here D is $D_1 + iD_2$, D_μ being the covariant derivative, $S = s_1 + s_2 + \dots + s_J$ and J is the number of Z 's. The string solution is in this case called the spinning closed string since the string is spinning in the AdS_5 space [22, 31]. We find for large J and finite $\alpha = S/J$

$$\frac{E - S - J}{\lambda'} = \frac{1 + \alpha}{\alpha} \frac{Jm^2}{2(1 + 1/J)} \left(1 - \frac{(1 + \alpha)}{\alpha} \frac{\pi^2 m^2}{3(J + 1)} \right) + \mathcal{O}(m^6) \quad (1.2)$$

This is consistent with previous results for spinning strings [28].³

The *orbifolded* circular string solution, that we also consider in this paper, is a generalization of the circular string solution to string theory on $\text{AdS}_5 \times S^5/\mathbb{Z}_M$ so that the subspace in which we have the string is $\mathbb{R} \times S^3/\mathbb{Z}_M$. The dual gauge theory is a $\mathcal{N} = 2$ quiver gauge theory (QGT) with the orbifolded circular string corresponding to a completely symmetrized trace of J_1 complex scalars Z and J_2 complex scalars X with $J_1 = J_2 = J/2$ and a suitably inserted twist matrix [34]. The Bethe ansatz that provides the anomalous dimension for these operators [35–37] contains a twist depending on the winding through the ratio m/M and its solution gives back the energy of the circular rotating string by setting $M = 1$. These operators cannot be directly inherited from the parent $\mathcal{N} = 4$ theory, in fact, because of the appearance of the twist matrix, the winding state involves the twisted sectors of the $\mathcal{N} = 2$ QGT. We generalize the solution of the Bethe equations for the $\alpha = 1/2$ circular string to the orbifolded case. This is readily achieved and the result is again (1.1) with the substitution of m with m/M .⁴

The *winding state* is given by the limit of orbifolded circular string solution with M^2/J fixed for $J \rightarrow \infty$ [34]. In [34] a Penrose limit of $\text{AdS}_5 \times S^5/\mathbb{Z}_M$ giving a pp-wave background with a compact spatial direction (with 24 supersymmetries) is considered. The winding state corresponds to a string winding around the compact spatial direction.⁵ One of the reasons why it is interesting to study this type of state is that, in the large M limit, the instabilities of the circular string are absent. Moreover since the winding state on the string side is a vacuum state for the string excitations, the energy found from the Bethe equations should be reproduced purely by quantum string effects on the string theory side.

³The problem of computing higher order finite size corrections has also been addressed in [32, 33]. It would be interesting to compare their results with ours.

⁴In [38] the $\mathfrak{su}(2)$ decoupling limit of [16–20] is generalized to orbifolded $\mathcal{N} = 2$ quiver gauge theory.

⁵The identification of the winding state studied in this paper and the winding state of [34] is provided in detail in [39].

In this paper we shall not only be concerned with solutions of the Bethe equations but we will also try to match the results obtained from the Bethe equations with those coming from the corresponding coherent state “Landau-Lifshitz” (LL) sigma model which describes low energy states of the ferromagnetic spin-chain [25–27]. This LL type action creates a connection between the gauge theory and the string theory pictures, suggesting how a continuous string action may appear from the gauge theory, as well as providing further evidence of the microscopic spin-chain description of string theory. The LL action is an effective low-energy action that arises from the gauge theory - spin chain and the quantum superstring, and, as such, it cannot be expected to lead to a well-defined quantum theory. However, supplemented with an appropriate regularization prescription and with higher-derivative counterterms, the LL model has been able to capture a non-trivial part of the quantum corrections to the “microscopic” theories, the string and spin-chain [23, 24].

We will compare up to the order $1/J^2$ the energy for the circular string and the winding state obtained from the Bethe equations with those derived from the LL model. For the first two leading terms in the winding number m , where the Bethe equations have been solved at any order in J , using, in the case of the winding state, an orbifolded version of the LL sigma-model, we will show that the results of the two computations actually match.

For the circular string the first order correction in $1/J$ can be reproduced by ζ -function regularization, as found in [28, 29, 23]. This matches the m^2 and m^4 corrections at first order in $1/J$ in (1.1). We go on to compute, in two different parametrizations of the LL model, the second order correction in $1/J$, again using ζ -function regularization. The two parametrizations, which give rise to rather different effective Lagrangians, yield in a non-trivial fashion the same result (this happens only thanks to a non-trivial cancellation of divergences).⁶ The result matches the m^2 part of (1.1), at second order in $1/J$, but, however, not the m^4 part. This can be explained by the fact that the regularization that actually corresponds to the UV finite microscopic theories not necessarily is the ζ -function regularization, as suggested also in [23, 24]. Clearly, the most satisfactory way to resolve this question would be to make a complete superstring calculation to this order, since the superstring sigma-model should automatically choose the right regularization prescription.

In the case of the winding state we consider the orbifolded LL model that arises either by taking a limit of the classical sigma-model on $\mathbb{R} \times S^3/\mathbb{Z}_M$, or from orbifolding the Bethe equations for the $\mathfrak{su}(2)$ sector. While the target space of the LL model is S^2 with rotation angle $\varphi = \phi_1 - \phi_2$, the orbifolded LL model corresponds to the same sigma-model action but with the identification $\varphi \equiv \varphi + 4\pi/M$. Taking the limit of large J , with M^2/J fixed, it reveals to leading order a cylinder $S^1 \times \mathbb{R}$ as the target space for the sigma-model. The winding state describes a string winding around the compact direction.

We are able to match the leading order, the $1/J$ correction and the $1/J^2$ correction to the energy of the winding state, as computed from the Bethe equations and the orbifolded LL model. To first order in $1/J$ we have a m^2/J term only. This is matched by observing that certain non-normal ordered terms in the $1/J$ Hamiltonian can contribute to the energy due to the absence of a zero mode for the string.

⁶The result differs from [23], see section 2.2.

At $1/J^2$ we have to use second order perturbation theory. This gives rise to two terms, an m^2/J^2 and an m^4/J^2 term. The matching of both these two terms is highly interesting. The m^2/J^2 term arises from the mean value of the interaction Hamiltonian on the winding state. It can be matched by carefully considering the ordering of the two coordinates that we use to parameterize the target space. We show that irrespective of what ordering convention we use, we always get the same answer for the m^2/J^2 term. In particular, one can use Weyl ordering [40]. The m^4/J^2 term comes instead by summing over the set of intermediate states. We find that the only non-zero contribution to the energy is found by summing over the individual contribution of all the possible two-oscillator string states created from the vacuum. We believe that this matching is rather novel and non-trivial in this respect, in that it is the first time that a successful match of gauge theory and a continuous sigma-model has relied on summing over intermediate states with a different number of oscillators compared to the external state.

This paper is organized as follows. In section 2 we consider finite-size corrections to the circular string state in the $\mathfrak{su}(2)$ sector and the spinning string in the $\mathfrak{sl}(2)$ sector. In section 2.1 we find the finite-size corrections (1.1) and (1.2) from the Bethe equations. We subsequently consider the finite-size corrections in the $\mathfrak{su}(2)$ sector as computed from the LL model in section 2.2. In section 3 we consider the finite-size size corrections to the winding state, first from the Bethe equations in section 3.1 and subsequently from the orbifolded LL model in section 3.2. We conclude and discuss future perspectives in section 4.

2. Finite-size corrections to the circular/spinning string state

Our aim here is to compute the quantum finite size corrections to the one-loop anomalous dimensions of operators of the form $\text{Tr}(Z^{J_1} X^{J_2})$ and $\text{Tr}(D^S Z^J)$ in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors, respectively. These are conjectured to be equal to the energy of the circular rotating string with two independent angular momenta J_1 and J_2 in S^5 , winding number m , and to the energy of the spinning string with spin S in AdS_5 and angular momentum J on S^5 . The latter solution may be viewed as an analytic continuation of the first so that the two cases can be treated simultaneously just by keeping track of the sign differences in the Bethe equations for the two sectors. Both the $\mathfrak{su}(2)$ sector and the $\mathfrak{sl}(2)$ sectors are described by a XXX Heisenberg spin chain, the $\mathfrak{su}(2)$ with spin $1/2$ and the $\mathfrak{sl}(2)$ with spin $-1/2$.

The Bethe equations will be solved by reformulating the problem in terms of the resolvent function $G(x)$ as in [41, 28] but by keeping into account also the so-called anomalous contribution arising from the fraction of the Bethe roots whose distance is of order $1/J$ [28, 42]. We shall explore a different region of the parameters compared to the one studied in [28]. With J large and $\alpha = K/J$ fixed but finite, (here K is the number of impurities *i.e.* J_2 for the $\mathfrak{su}(2)$ sector and S for the $\mathfrak{sl}(2)$ sector) we shall be able to go beyond the $1/J$ order result of [28] and determine the spectrum for any value of J in an expansion in the winding number m . This will be done by taking advantage of an exact property of the zeroes of the Laguerre polynomial found in [30].

2.1 All-order finite-size effects from Bethe equations

The spectrum of anomalous dimension of operators in the $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ sectors is given, at one-loop, by the solution of the Bethe equations [4]

$$\left(\frac{u_k + \frac{i}{2}r}{u_k - \frac{i}{2}r}\right)^J = \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j - i} \quad (2.1)$$

where $r = -1$ for $\mathfrak{sl}(2)$ and $r = 1$ for $\mathfrak{su}(2)$. The indices j and k go from 1 to K , K being the magnon number. The Bethe equations describe completely the spectrum of the spin chain and for $r = 1$ are those of the Heisenberg magnet.

If we take the logarithm of (2.1), we get⁷

$$\pi n + rJ \arctan\left(\frac{1}{2u_k}\right) = \sum_{j \neq k} \arctan\left(\frac{1}{u_k - u_j}\right) \quad (2.2)$$

where $n \in \mathbb{Z}$ reflects the arbitrariness in choosing the branch of the logarithm. In general one can choose a different n for each k , but we restrict ourselves here to the special case where n is the same for all k . The constraint from the cyclicity of the trace gives the momentum condition

$$m \equiv \frac{1}{\pi} \sum_{k=1}^K \arctan\left(\frac{1}{2u_k}\right) \in \mathbb{Z} \quad (2.3)$$

The one-loop contribution to the energy is

$$\mathcal{E} = \frac{\lambda}{8\pi^2} \sum_k \frac{1}{u_k^2 + \frac{1}{4}} \quad (2.4)$$

where $\mathcal{E} = E - J$ for the $\mathfrak{su}(2)$ sector and $\mathcal{E} = E - S - J$ for the $\mathfrak{sl}(2)$ sector, with E being the full scaling dimension of the operator. In the $\mathfrak{su}(2)$ sector $J = J_1 + J_2$ where J_1 and J_2 are two of the R-charges. In the $\mathfrak{sl}(2)$ sector J is an R-charge while S is an angular momentum. Summing over all k in (2.2) gives zero on the right-hand side. Therefore, we get the constraint

$$Kn + rJm = 0 \quad (2.5)$$

This is as in string theory, it provides the level matching condition in the presence of a winding. We define α as

$$\alpha \equiv \frac{K}{J} \quad (2.6)$$

Therefore $n = -rm/\alpha$. The string requires both n and m to be integers so that α can only be a divisor of m , for the spin-chain we can instead consider states with any value of α .

It is apparent that the one-loop contribution to the energy (2.4) of the state defined by the Bethe equations (2.2) and the momentum constraint (2.3) has the functional form $\mathcal{E} = \lambda f(J, \alpha, m)$. The full function $f(J, \alpha, m)$ is unknown. In previous works on the subject [41, 28], $f(J, \alpha, m)$ has been expanded as

$$f(J, \alpha, m) = \frac{1}{J} f_1(\alpha, m) + \frac{1}{J^2} f_2(\alpha, m) + \frac{1}{J^3} f_3(\alpha, m) + \dots \quad (2.7)$$

⁷Here and in the following we are considering the branch of \arctan with $\arctan(0) = 0$.

In this approach $f_1(\alpha, m)$ and $f_2(\alpha, m)$ have been found, revealing the following expression for the one-loop contribution to the energy

$$\mathcal{E} = \frac{\lambda' J \hat{m}^2}{2} + r \frac{\lambda' \hat{m}^2}{2} + \frac{\lambda'}{2} \sum_{n=1}^{\infty} \left[n^2 \sqrt{1 - r \frac{4\hat{m}^2}{n^2}} - n^2 + 2r\hat{m}^2 \right] + \mathcal{O}(\lambda' J^{-1}) \quad (2.8)$$

where we defined

$$\hat{m} \equiv \sqrt{\frac{1 - r\alpha}{\alpha}} m \quad (2.9)$$

In the following we propose instead to consider the expansion

$$f(J, \alpha, m) = m^2 g_1(J, \alpha) + m^4 g_2(J, \alpha) + m^6 g_3(J, \alpha) + \dots \quad (2.10)$$

Using this way of expanding $f(J, \alpha, m)$ we will be able to determine $g_1(J, \alpha)$ and $g_2(J, \alpha)$. However, as we shall see below, the result is only reliable for $K = \alpha J \gg m$ since otherwise one cannot make sense of this expansion.

Obviously (2.10) is only an expansion in a formal sense since if for example one sets $m = 1$ then one needs all the infinite number of terms in the expansion (2.10). The expansion is nevertheless still useful since one can keep m free. Moreover, in section 3.1 we shall show that in the orbifolded theory only a finite number of terms in the expansion (2.10) contributes to a given power of $1/J$ in a $1/J$ expansion, thus making it sensible also to set $m = 1$.

Define now

$$x_k \equiv \frac{2\pi r n}{J} u_k = -\frac{2\pi m}{\alpha J} u_k \quad (2.11)$$

and the expansion parameter ϵ

$$\epsilon \equiv \frac{4\pi^2 m^2}{\alpha^2 J^2} = \frac{4\pi^2 m^2}{K^2} \quad (2.12)$$

which is small, $\epsilon \ll 1$, only for $K = \alpha J \gg m$, *i.e.* we need a large magnon number.

Then (2.2) can be expanded in ϵ as

$$1 + \frac{1}{x_k} - \frac{\epsilon}{12x_k^3} + \frac{\epsilon^2}{80x_k^5} + \dots = \frac{r}{J} \sum_{j \neq k} \left(\frac{2}{x_k - x_j} - \frac{2\epsilon}{3(x_k - x_j)^3} + \frac{2\epsilon^2}{5(x_k - x_j)^5} + \dots \right) \quad (2.13)$$

We furthermore have that the momentum constraint (2.3) is expanded as

$$\sum_k \left(\frac{1}{x_k} - \epsilon \frac{1}{12x_k^3} + \dots \right) = -\alpha J \quad (2.14)$$

Define the resolvent

$$G(x) = \frac{1}{J} \sum_{k=1}^K \frac{1}{x - x_k} \quad (2.15)$$

We expand $G(x)$ in powers of ϵ as

$$G(x) = G_0(x) + \epsilon G_1(x) + \epsilon^2 G_2(x) + \dots \quad (2.16)$$

The momentum constraint to first order in ϵ gives

$$G_0(0) = \alpha, \quad G_1(0) = \frac{1}{24}G_0''(0) \quad (2.17)$$

To first order in ϵ we can write the Bethe equations (2.2) as

$$\begin{aligned} rG(x)^2 + \frac{r}{J}G'(x) &= -\frac{\alpha}{x} + \left(1 + \frac{1}{x} - \frac{\epsilon}{12x^3}\right)G(x) + \frac{\epsilon}{12}\left(\frac{\alpha}{x^3} + \frac{G'(0)}{x^2}\right) \\ &+ r\frac{2\epsilon}{3J^2}\sum_k \frac{1}{x-x_k} \sum_{j \neq k} \frac{1}{(x_k-x_j)^3} + \mathcal{O}(\epsilon^2) \end{aligned} \quad (2.18)$$

Using (2.16) we see that $G_0(x)$ should obey the equation

$$rG_0(x)^2 + \frac{r}{J}G_0'(x) = -\frac{\alpha}{x} + \left(1 + \frac{1}{x}\right)G_0(x) \quad (2.19)$$

Introducing the function

$$Q(x) = \prod_{i=1}^K (x-x_i) \quad (2.20)$$

known as the eigenvalue of the Baxter Q -operator, we can write now

$$G(x) = \frac{1}{J} \frac{Q'(x)}{Q(x)} \quad (2.21)$$

Then, using (2.16) at the lowest order in ϵ , (2.19) is equivalent to

$$\left[x \frac{d^2}{dx^2} - rJ(x+1) \frac{d}{dx} + r\alpha J^2 \right] Q = 0 \quad (2.22)$$

It is useful to define the variable $y \equiv rJx$. We see that (2.22) written in terms of y is equivalent to the Laguerre differential equation (A.1) given in appendix A with $\nu = -rJ-1$ and $\lambda = \alpha J$. This means that we have the solution

$$Q(x) \propto L_{\alpha J}^{-rJ-1}(rJx) \quad (2.23)$$

where $L_{\lambda}^{\nu}(y)$ is the Laguerre polynomial, see appendix A. We can now use the sum rules (A.2) and (A.3) for the zeroes of Laguerre polynomials [30]. We label the zeroes as y_k , $k = 1, 2, \dots, K$. It is not difficult to see that (A.2) is equivalent to the zeroth order part of (2.13), *i.e.*

$$\frac{2r}{J} \sum_{j \neq k} \frac{1}{x_k - x_j} = 1 + \frac{1}{x_k} \quad (2.24)$$

when setting $y_k = rJx_k$. Thus we have a clear connection between the Bethe roots x_k and the zeroes of an associated Laguerre polynomial. Instead the sum (A.3) gives

$$\sum_{j \neq k} \frac{1}{(x_k - x_j)^3} = -\frac{J(J-2r)}{8x_k^3} - \frac{J^2(1-2\alpha r)}{8x_k^2} \quad (2.25)$$

Using this in (2.18) one obtains the following equation for $G_1(x)$

$$\begin{aligned} & [2rxG_0(x) - x - 1]G_1(x) + \frac{rx}{J}G_1'(x) \\ &= -\frac{1}{12} \left(\frac{rJ-1}{x^2} + \frac{rJ(1-2\alpha r)}{x} \right) [G_0(x) - \alpha - xG_0'(0)] + \frac{rJ-2}{24}G_0''(0) \end{aligned} \quad (2.26)$$

The energy \mathcal{E} is computed using

$$\sum_k \frac{1}{u_k^2 + \frac{1}{4}} = -\epsilon J G_0'(0) + \epsilon^2 J \left[-G_1'(0) + \frac{1}{24}G_0'''(0) \right] + \mathcal{O}(\epsilon^3) \quad (2.27)$$

We see that all we need to know in order to find the energy is $G_0'(0)$, $G_0'''(0)$ and $G_1'(0)$. $G_0'(0)$ and $G_0'''(0)$ are found by writing $G_0(x)$ in a Taylor expansion

$$G_0(x) = \frac{1}{2} + G_0'(0)x + \frac{1}{2}G_0''(0)x^2 + \frac{1}{6}G_0'''(0)x^3 + \dots \quad (2.28)$$

Inserting this into (2.19) and expanding in x we find

$$G_0'(0) = -\frac{(1-r\alpha)\alpha J}{J-r}, \quad G_0''(0) = \frac{2\alpha J^2(1-r\alpha)(1-2r\alpha)}{(J-r)(J-2r)} \quad (2.29)$$

$$G_0'''(0) = -\frac{6\alpha J^3(1-r\alpha)[J(1-5r\alpha+5\alpha^2) - r + 6\alpha - 6r\alpha^2]}{(J-r)^2(J-2r)(J-3r)} \quad (2.30)$$

Making a similar Taylor expansion for $G_1(x)$ as in (2.28) we get from (2.26)

$$G_1'(0) = \frac{Jr}{72}[G_0'''(0) + (3-6r\alpha)G_0''(0)] \quad (2.31)$$

Using this in (2.27) we get

$$\mathcal{E} = \frac{\lambda' J^2 \hat{m}^2}{2(J-r)} - \frac{\pi^2 \lambda' J^2 \hat{m}^4}{6(J-r)^2} + \mathcal{O}(m^6) \quad (2.32)$$

where \hat{m} is defined in (2.9). We see thus that we have obtained $g_1(J, \alpha)$ and $g_2(J, \alpha)$ in the formal expansion (2.10) in powers of m , as promised. It is useful to recall here the validity of this equation. We chose an expansion parameter ϵ defined in (2.12), which, in order to be small, requires a large number of impurities $K \gg m$. At the same time we expanded in the variable $m = -ran$ so that α has to be kept finite, it cannot be sent to infinity for example as in the case studied in [28]. However the coefficient of the m^2 and m^4 terms $g_1(J, \alpha)$ and $g_2(J, \alpha)$ for fixed finite α provide the finite size corrections to all orders in J . In the next section, using the Landau-Lifshitz model, we will be able to match, using this formula, the \hat{m}^2/J^2 term which could not be derived in [28]. Expanding (2.32) we get

$$\mathcal{E} = \lambda' J \left(\frac{\hat{m}^2}{2} + r \frac{\hat{m}^2}{2J} + \frac{\hat{m}^2}{2J^2} - \frac{\pi^2 \hat{m}^4}{6J} - r \frac{\pi^2 \hat{m}^4}{3J^2} \right) + \mathcal{O}(m^6) + \mathcal{O}(\lambda' J^{-2}) \quad (2.33)$$

We can now compare the result (2.32) with the result (2.8) for the one-loop energy contribution obtained in [28] in the $1/J$ expansion (2.7). Expanding (2.8) in powers of \hat{m} , we obtain

$$\mathcal{E} = \lambda' J \left(\frac{\hat{m}^2}{2} + r \frac{\hat{m}^2}{2J} - \zeta(2) \frac{\hat{m}^4}{J} - 2r\zeta(4) \frac{\hat{m}^6}{J} \right) + \mathcal{O}(m^8) + \mathcal{O}(\lambda' J^{-1}) \quad (2.34)$$

which is seen to match (2.33), for the common terms.

2.2 Results from Landau-Lifshitz sigma-model

A convenient framework where to compute finite size corrections to the energy of a given state is provided by the Landau-Lifshitz (LL) model [25–27].

The LL model was introduced in the study of the low energy spectrum of the ferromagnetic Heisenberg spin chain. The Sigma-model action is given by

$$I = \frac{\lambda' J}{4\pi} \int dt \int_0^{2\pi} d\sigma \left[\vec{C}(\vec{n}) \cdot \dot{\vec{n}} - \frac{1}{4} (\vec{n}')^2 \right] \quad (2.35)$$

where \vec{n} is a three-dimensional unit-vector parameterizing the two-sphere, the “prime” means derivative with respect to σ and the “dot” derivative with respect to t . Note that the first term in (2.35) is a Wess-Zumino type term which is proportional to the area spanned between the trajectory and the north pole of the two-sphere [25]. Choosing the parametrization

$$\vec{n} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta) \quad (2.36)$$

we have that

$$\vec{C}(\vec{n}) \cdot \dot{\vec{n}} = \sin \theta \dot{\varphi}, \quad (\vec{n}')^2 = (\theta')^2 + \cos^2 \theta (\varphi')^2 \quad (2.37)$$

The action (2.35) corresponds to the Hamiltonian

$$H = \frac{\lambda' J}{4\pi} \int_0^{2\pi} d\sigma \frac{1}{4} (\vec{n}')^2 \quad (2.38)$$

Classically, a circular rotating string is a configuration of the type

$$\theta = 0, \quad \varphi = 2m\sigma \quad (2.39)$$

where m is an integer winding number. We want to study fluctuations around this solution. This LL solution corresponds to the term of order λ in the circular string solution [3, 31]. The classical energy, obtained by expanding the energy of the full solution, $E = \sqrt{J^2 + \lambda m^2}$, is given by

$$E_0 = J \left(1 + \lambda' \frac{m^2}{2} + \mathcal{O}(\lambda'^2) \right) \quad (2.40)$$

The sigma-model action as derived from the Heisenberg spin chain also contains higher derivative terms coming in at higher powers in $1/J$. Adding the first higher-derivative term, the Hamiltonian (2.38) changes to

$$H = \frac{\lambda' J}{4\pi} \int_0^{2\pi} d\sigma \left[\frac{1}{4} (\vec{n}')^2 - \frac{\pi^2}{12J^2} (\vec{n}'')^2 + \mathcal{O}(J^{-4}) \right] \quad (2.41)$$

We record that

$$\begin{aligned} (\vec{n}'')^2 &= (\theta'')^2 + \cos^2 \theta (\varphi'')^2 + (\theta')^4 + \cos^2 \theta (\varphi')^4 + 2(2 - \cos^2 \theta) (\theta')^2 (\varphi')^2 \\ &\quad + 2 \sin \theta \cos \theta [\theta'' (\varphi')^2 - 2\theta' \varphi' \varphi''] \end{aligned} \quad (2.42)$$

Evaluated on the classical state (2.39) the Hamiltonian (2.41) gives

$$H = \frac{\lambda' J}{4\pi} \int_0^{2\pi} d\sigma \left[\frac{1}{4} (\varphi')^2 - \frac{\pi^2}{12J^2} (\varphi')^4 + \mathcal{O}(J^{-4}) \right] = \frac{\lambda' J m^2}{2} - \frac{2\pi^2 \lambda' m^4}{3J} + \mathcal{O}(\lambda' J^{-3}) \quad (2.43)$$

The first term reproduces the classical energy at this order in λ' , (2.40), the second term has to be viewed as a quantum counterterm added in order to match the discrete spin chain result. We see that the higher derivative terms start contributing at the order λ'/J .

To study the fluctuations we denote the ground state for this LL model as $|0_m\rangle$. We want to compute quantum corrections to the ground state energy. To do this, we expand the LL action (2.35) around the solution (2.39) and then we quantize the Hamiltonian for the fluctuations. We write this as

$$\varphi = 2m\sigma + \frac{2f}{\sqrt{J}}, \quad \sin\theta = \frac{2g}{\sqrt{J}} \quad (2.44)$$

Here we parameterized the fluctuations using the functions $f(t, \sigma)$ and $g(t, \sigma)$ corresponding to the two spherical angles on the two-sphere. In appendix B we consider another parametrization of the two-sphere leading to equivalent results. The parametrization (2.44) gives the Lagrangian

$$J\mathcal{L} = 4gf\dot{\ } - \mathcal{H}_2 - \frac{1}{\sqrt{J}}\mathcal{H}_3 - \frac{1}{J}\mathcal{H}_4 + \dots \quad (2.45)$$

with

$$\mathcal{H}_2 = (f')^2 + (g')^2 - 4m^2g^2 \quad (2.46)$$

$$\mathcal{H}_3 = -8mf'g^2 \quad (2.47)$$

$$\mathcal{H}_4 = 4g^2(g'^2 - f'^2) \quad (2.48)$$

The Hamiltonian is given by

$$H = \frac{\lambda'}{4\pi} \int_0^{2\pi} d\sigma \left(\mathcal{H}_2 + \frac{1}{\sqrt{J}}\mathcal{H}_3 + \frac{1}{J}\mathcal{H}_4 + \dots \right) \quad (2.49)$$

The equations of motion are [23]

$$\dot{f} = -\frac{1}{2}(g'' + 4m^2g), \quad \dot{g} = \frac{1}{2}f'' \quad (2.50)$$

The solution can be written as

$$\begin{aligned} f(\tau, \sigma) &= \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \sqrt{w_n} (a_n e^{-i\omega_n \tau + i n \sigma} + a_n^\dagger e^{i\omega_n \tau - i n \sigma}) \\ g(\tau, \sigma) &= -\frac{i}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{\sqrt{w_n}} (a_n e^{-i\omega_n \tau + i n \sigma} - a_n^\dagger e^{i\omega_n \tau - i n \sigma}) \end{aligned} \quad (2.51)$$

where

$$\omega_n = \frac{1}{2}n\sqrt{n^2 - 4m^2}, \quad w_n = \sqrt{1 - \frac{4m^2}{n^2}}, \quad n = \pm 1, \pm 2, \dots \quad (2.52)$$

We impose the commutation relations

$$[f(\tau, \sigma), f(\tau, \sigma')] = 0, \quad [g(\tau, \sigma), g(\tau, \sigma')] = 0, \quad [f(\tau, \sigma), g(\tau, \sigma')] = i\pi\delta(\sigma - \sigma') - \frac{i}{2} \quad (2.53)$$

where the zero mode contribution has been subtracted. These give

$$[a_n, a_k^\dagger] = \delta_{n-k} \tag{2.54}$$

Note that with this choice of coordinates that parametrize the LL Lagrangian, even if f and g do not commute, (2.53), there are no ordering problems in the definitions of (2.47) and (2.48), g in fact commutes with f' .

As it is well-known [3, 31, 23], the solution (2.51) has unstable fluctuation modes with $n = \pm 1, \dots, \pm 2m$. They are a manifestation of an instability of the full homogeneous string solution. We will ignore these instabilities. Let us just comment that in the case of the winding state which will be considered in the next section and which is obtained by replacing m with $\frac{m}{M}$, there are no unstable modes or instabilities. More details can be found in the next section.

Another way to avoid instabilities is to consider, instead of the $\mathfrak{su}(2)$ sector discussed in this section, the stable solution of the $\mathfrak{sl}(2)$ sector, where one of the angular momenta is on AdS_5 and the other one is on S^5 [31, 43–45].

Leading correction to the circular string solution

Using the solutions (2.51), the quadratic Hamiltonian H_2 , which gives the leading correction to the energy of the ground state $|0_m\rangle$, can be written as

$$H_2 = \frac{\lambda'}{4\pi} \int_0^{2\pi} d\sigma \mathcal{H}_2 = \frac{\lambda'}{2} \sum_{n=-\infty}^{\infty} |\omega_n| (a_n a_n^\dagger + a_n^\dagger a_n) \tag{2.55}$$

and we then get

$$E_1 = \langle 0_m | H_2 | 0_m \rangle = \frac{\lambda'}{2} \sum_{n=-\infty}^{\infty} |\omega_n| \tag{2.56}$$

We see that the sum in the previous expression is divergent and needs to be regularized. A natural regularization choice is to subtract and add the divergent contribution and then, for the latter, to use the ζ -function regularization. We obtain [23]

$$E_1 = \frac{\lambda'}{2} \left[m^2 + \sum_{n=1}^{\infty} (n\sqrt{n^2 - 4m^2} - n^2 + 2m^2) \right] \tag{2.57}$$

This result is in agreement with the full string theory 1-loop computation [46–48] and with the Bethe ansatz computation on the spin chain side [28, 49]. We see that (2.57) expanded in powers of m , up to and including the m^4 term, agrees with (2.32) for the leading terms in a large J expansion.

Next to leading correction to the circular string solution

Now we compute the next subleading correction to the energy of the circular string. This corresponds to the λ'/J result obtained from the Bethe ansatz and to the 2-loop correction computed on the string theory side. We use standard perturbation theory. Since

$\langle 0_m | H_3 | 0_m \rangle = 0$, the first non trivial correction to the energy of the ground state is given by the second order perturbation theory formula

$$E_2 = \sum_{|i\rangle \neq |0_m\rangle} \frac{|\langle 0_m | H_3 | i \rangle|^2}{E_{0_m} - E_i} + \langle 0_m | H_4 | 0_m \rangle = E_2^a + E_2^b \quad (2.58)$$

where $|i\rangle$ is an intermediate state.

We start by computing E_2^a . We have that

$$H_3 = -\frac{\lambda'}{\pi\sqrt{J}} m \int_0^{2\pi} d\sigma 2f' g^2 \quad (2.59)$$

We want to compute $\langle 0_m | H_3 | i \rangle$, where the intermediate state $|i\rangle$ is given by

$$|i\rangle = \frac{1}{\sqrt{\mathcal{N}}} a_l^\dagger a_p^\dagger a_q^\dagger |0_m\rangle \quad (2.60)$$

where \mathcal{N} is a normalization constant. We see that it is necessary to chose intermediate states for which the number of oscillators is different from the one of the external state. The relevance of keeping into account the contributions coming from this channel was stressed not only in ref. [23], in a similar LL type calculation, but also in the pp-wave string field theory context in refs. [50, 51]. We get

$$\begin{aligned} E_2^a &= -\frac{1}{3!} \sum_{l \neq p \neq q} \frac{|\langle 0_m | H_3 a_l^\dagger a_p^\dagger a_q^\dagger | 0_m \rangle|^2}{\lambda'(|\omega_l| + |\omega_p| + |\omega_q|)} - \frac{1}{2} \sum_{l \neq p} \frac{|\langle 0_m | H_3 a_l^\dagger a_p^\dagger a_p^\dagger | 0_m \rangle|^2}{\lambda'(|\omega_l| + 2|\omega_p|)} \\ &= -\frac{m^2 \lambda'}{3J} \sum_{l \neq p \neq q} \frac{\left(l \sqrt{\frac{w_l}{w_p w_q}} + p \sqrt{\frac{w_p}{w_l w_q}} + q \sqrt{\frac{w_q}{w_l w_p}} \right)^2}{l^2 w_l + p^2 w_p + q^2 w_q} - \frac{m^2 \lambda'}{2J} \sum_{l \neq p} \frac{\left(l \sqrt{\frac{w_l}{w_p^2}} + 2p \sqrt{\frac{1}{w_l}} \right)^2}{l^2 w_l + 2p^2 w_p} \end{aligned} \quad (2.61)$$

where the sums go from $-\infty$ to ∞ . Here we have separated the contribution obtained when l, p, q are all different, which is given by the first term on the r.h.s. of (2.61) both in the first and the second line, and the contribution obtained when two integers among l, p, q are equal, which is given by the second term of (2.61).

We now proceed to compute E_2^b . We have that

$$\begin{aligned} H_4 &= \frac{\lambda'}{4\pi J} \int_0^{2\pi} d\sigma [4g^2(g'^2 - f'^2)] = -\frac{\lambda'}{J} \sum_{l,p,q,s=-\infty}^{\infty} \frac{lp}{8\sqrt{w_l w_p w_q w_s}} \delta(l+p+q+s) \\ &\times \left[(1+w_l w_p) (a_s a_q a_{-l}^\dagger a_{-p}^\dagger + a_{-s}^\dagger a_{-q}^\dagger a_l a_p + a_s a_q a_l a_p + a_{-s}^\dagger a_{-q}^\dagger a_{-l}^\dagger a_{-p}^\dagger \right. \\ &\quad \left. - a_s a_{-q}^\dagger a_l a_p - a_s a_{-q}^\dagger a_{-l}^\dagger a_{-p}^\dagger - a_{-s}^\dagger a_q a_l a_p - a_{-s}^\dagger a_q a_{-l}^\dagger a_{-p}^\dagger) + (1-w_l w_p) \right. \\ &\quad \left. \times \left(a_{-s}^\dagger a_q a_{-l}^\dagger a_p + a_s a_{-q}^\dagger a_{-l}^\dagger a_p + a_{-s}^\dagger a_q a_l a_{-p}^\dagger + a_s a_{-q}^\dagger a_l a_{-p}^\dagger - a_s a_q a_{-p}^\dagger a_p - a_s a_q a_l a_{-p}^\dagger \right) \right] \end{aligned} \quad (2.62)$$

We get⁸

$$\langle 0_m | H_4 | 0_m \rangle = \frac{\lambda'}{8J} \sum_{l,p=-\infty}^{\infty} l^2 \left(\frac{1}{w_l w_p} - \frac{w_l}{w_p} \right) \quad (2.63)$$

⁸The same computation has been done in [23]. However our result for $\langle 0_m | H_4 | 0_m \rangle$ differs from the one of [23]. We show in appendix B that, using a different parametrization, we obtain the same result for $\langle 0_m | H_4 | 0_m \rangle$ as obtained in this section. Moreover our result expanded in m^2 reproduces exactly the result (2.32) derived from the Bethe ansatz.

The divergent sums in the previous expression can be regularized as before by adding and subtracting the divergent contribution and then using ζ -function regularization [23]. The result is

$$E_2^b = \langle 0_m | H_4 | 0_m \rangle = \frac{\lambda'}{4J} \left[\sum_{l=1}^{\infty} \left(\frac{l^2}{w_l} - l^2 w_l - 4m^2 \right) - 2m^2 \right] \left[2 \sum_{p=1}^{\infty} \left(\frac{1}{w_p} - 1 \right) - 1 \right] \quad (2.64)$$

The total energy at this order is then given by the sum of (2.43), (2.57), (2.61) and (2.64) and has the form [23]

$$E = J \left[1 + \frac{m^2 \lambda'}{2} \left(1 + \frac{c_1}{J} + \frac{c_2}{J^2} + \mathcal{O} \left(\frac{1}{J^3} \right) \right) + \mathcal{O}(\lambda'^2) \right] \quad (2.65)$$

The coefficients c_1 and c_2 given by regularized sums may be evaluated numerically as in [23]. Taking $m = 1$ (and ignoring imaginary contributions of unstable modes) we get⁹

$$c_1 = -0.893, \quad c_2 = -5.44 \quad (2.66)$$

We now want to compare the result for E_2 with the Bethe ansatz result (2.32). For this case, being the angular momenta J_1 and J_2 equal, we have to set $\alpha = 1/2$ and $r = 1$ so that $\hat{m} = m$. As before, we expand the energy in powers of m^2 . We have

$$E_2 = \frac{\lambda'}{J} \left(\frac{m^2}{2} - \frac{2}{3} \pi^2 m^4 \right) + \mathcal{O}(m^6) + \mathcal{O}(\lambda' J^{-2}) \quad (2.67)$$

Putting together all the contributions, including the higher derivative term appearing in (2.43) up to the order m^4 , we obtain the following expression for the energy of the circular string

$$E = J \left[1 + \lambda' \left(\frac{m^2}{2} + \frac{m^2}{2J} + \frac{m^2}{2J^2} - \frac{\pi^2 m^4}{6J} - \frac{4\pi^2 m^4}{3J^2} \right) \right] + \mathcal{O}(m^6) + \mathcal{O}(\lambda' J^{-2}) \quad (2.68)$$

We see that this result coincides exactly with the one obtained from the Bethe ansatz, eq. (2.33), for the terms proportional to m^2 .

The coefficient of the m^4/J term is also reproduced but from this LL computation we find a mismatch for the coefficient of the term proportional to m^4/J^2 . This discrepancy is probably due to the fact that the ζ -function regularization is not appropriate at the order $1/J^2$ (see also [23] for comments on this point).

Let us finally comment that in appendix B the same computation is performed using a different parametrization and it is shown that, using the same procedure described above, we get again the result (2.68). This result arises also thanks to a non-trivial cancelation of divergences.

⁹The difference in the value of c_2 compared to the one obtained in [23] is due to the different form for the term (2.64) (see the previous footnote), the normalization of the intermediate states (2.60) and the inclusion of the higher derivative term (2.43).

3. Finite-size corrections to the winding state

In section 2 we considered the finite-size effects for the spinning/circular string state both from the Bethe equations and from the Landau-Lifshitz model point of view. In this section we generalize the circular string state to an orbifolded circular string state. Geometrically, this means that whereas before we were considering a string winding around an S^3 we consider instead now a string winding around S^3/\mathbb{Z}_M . For $M = 1$ it reduces to the circular string state considered in section 2. Instead when considering a limit of large M and large J , such that M^2/J is finite, the orbifolded circular string state becomes a winding state. This winding state is related to a winding state for a particular pp-wave background considered in [34] (see also [39]).

In section 3.1 we introduce the orbifolded circular string state and we generalize the result of section 2.1 on the Bethe equations to this case. Then in section 3.2 we consider the orbifolded Landau-Lifshitz model and we match the finite-size corrections to the energy found from the Bethe equation to order $1/J^2$.

3.1 Winding state from Bethe equations

In this section we generalize the result of section 2.1 on the Bethe equations to the case of the orbifolded circular string state.

$\mathcal{N} = 2$ Orbifolded quiver gauge theory. Consider the $\mathbb{C}^2/\mathbb{Z}_M \times \mathbb{C}$ orbifold defined by the identification

$$(Z, X, W) \equiv (\theta Z, \theta^{-1} X, W) \tag{3.1}$$

where Z , X and W are three complex scalars and we define

$$\theta \equiv \exp\left(\frac{2\pi i}{M}\right) \tag{3.2}$$

If we place N coincident D3-branes at the orbifold singularity of the $\mathbb{C}^2/\mathbb{Z}_M \times \mathbb{C}$ orbifold we get a correspondence between a four-dimensional $\mathcal{N} = 2$ quiver gauge theory, which we here dub the $\mathcal{N} = 2$ orbifolded quiver gauge theory, and type IIB string theory on $\text{AdS}_5 \times S^5/\mathbb{Z}_M$ [52].

$\mathcal{N} = 2$ orbifolded quiver gauge theory consists of M vector multiplets and M hypermultiplets. Thus, the gauge group of the $\mathcal{N} = 2$ orbifolded quiver gauge theory is M products of $U(N)$, one of each node in the quiver. The easiest way to realize $\mathcal{N} = 2$ orbifolded quiver gauge theory is to start with $\mathcal{N} = 4$ SYM theory with gauge group $U(NM)$, and then do a \mathbb{Z}_M projection. The surviving components of the scalar fields are $N \times N$

matrices which are embedded in $MN \times MN$ $\mathcal{N} = 4$ variables as follows

$$\begin{aligned}
 Z &= \begin{pmatrix} 0 & Z_1 & & & \\ & 0 & Z_2 & & \\ & & & \ddots & \\ & & & 0 & Z_{M-1} \\ Z_M & & & & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & & & & X_M \\ X_1 & 0 & & & \\ & X_2 & 0 & & \\ & & & \ddots & \\ & & & & X_{M-1} & 0 \end{pmatrix}, \\
 W &= \begin{pmatrix} W_1 & & & & \\ & W_2 & & & \\ & & \ddots & & \\ & & & & W_M \end{pmatrix}
 \end{aligned} \tag{3.3}$$

In particular, we can consider this projection in the $\mathfrak{su}(2)$ sector of $U(NM)$ $\mathcal{N} = 4$ SYM theory. In $U(NM)$ $\mathcal{N} = 4$ SYM theory any single-trace operator in the $\mathfrak{su}(2)$ sector (containing the scalars Z and X) can be gotten from

$$\text{Tr}(Z^{J_1} X^{J_2}) \tag{3.4}$$

by permuting the X 's and Z 's (and making linear combination). After the Z_M projection a single-trace operator should instead be gotten from

$$\text{Tr}(S^m Z^{J_1} X^{J_2}) \tag{3.5}$$

with the quantization condition

$$\frac{J_1 - J_2}{M} \in \mathbb{Z} \tag{3.6}$$

and where $k = 0, 1, \dots, M - 1$ and S is the twist matrix defined as

$$S = \text{diag}(\theta, \theta^2, \dots, \theta^M) \tag{3.7}$$

with θ given by (3.2). The scalars Z and X obey the following relations

$$ZS = \theta SZ, \quad XS = \theta^{-1}SX \tag{3.8}$$

Bethe equations. Define the operator S_- by $S_-Z = X$ and $S_-X = 0$. Then we can write a general operator with K impurities as

$$\mathcal{O} \equiv \sum_{l_1 < l_2 < \dots < l_K} \Psi_{l_1, l_2, \dots, l_K} S_-^{l_1} S_-^{l_2} \dots S_-^{l_K} \text{Tr}(S^m Z^J) \tag{3.9}$$

where S_-^l acts on the l 'th Z in Z^J , *i.e.* the sites in Z^J run from site number 1 to J .

Just as in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [53, 5], the computation of dimensions of the operators of interest to us can be elegantly summarized by the action of an effective Hamiltonian. The $\mathcal{N} = 4$ dilatation operator is known explicitly in terms of its action on fields up to two loop order, and implicitly to three loop order [5, 54, 55]. That part which is known explicitly can be projected, using the orbifold projection, to

obtain a dilatation operator for the $\mathcal{N} = 2$ theory. Here, we shall be interested in computing dimensions of operators in the scalar $\mathfrak{su}(2)$ sector, so we only retain the parts of the operator which will contribute there. They can be obtained by simply substituting the matrices (3.3) into the F -terms of the $\mathcal{N} = 4$ operator, namely the relevant dilatation operator on states of the form (3.9) is

$$D_{1\text{-loop}} = -\frac{g_{\text{Y.M.}}^2}{8\pi^2} \text{Tr} \left(|[Z, X]|^2 \right) \quad (3.10)$$

where $g_{\text{Y.M.}}$ is the Yang-Mills coupling constant. This operator acts on nearest neighbors either as a permutation or as the identity operator and, due to the twist matrix S (3.7) contained in (3.9), the one-loop Hamiltonian (3.10) can be regarded as that of a Heisenberg $XXX_{1/2}$ spin-chain with twisted boundary conditions [35–37, 56]. The Bethe equations are then

$$\theta^{-2m} \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^J = \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j - i} \quad (3.11)$$

with the momentum constraint

$$\exp \left\{ 2i \sum_{k=1}^K \arctan \left(\frac{1}{2u_k} \right) \right\} = \theta^m \quad (3.12)$$

The dispersion relation is still given by (2.4).

We now introduce the *orbifolded circular string state*. Taking the logarithm of the Bethe equations (3.11) we get¹⁰

$$-\frac{2\pi m}{M} + J \arctan \left(\frac{1}{2u_k} \right) = \sum_{j \neq k} \arctan \left(\frac{1}{u_k - u_j} \right) \quad (3.13)$$

In general one can freely add a term πn_k to this, $n_k \in \mathbb{Z}$ for each impurity, corresponding to different branches of the logarithm in (3.11). However, for the orbifolded circular string state we consider the specific case in which all $n_k = 0$. Due to the momentum constraint (3.12) the logarithm of the constraint is taken to be

$$2 \sum_{k=1}^K \arctan \left(\frac{1}{2u_k} \right) = \frac{2\pi m}{M} \quad (3.14)$$

Summing over all k in (3.13) we get zero on the right-hand side. Combining this with (3.14) we obtain the constraint

$$K = \frac{J}{2} \quad (3.15)$$

Thus, we are considering a state with an equal number of X 's and Z 's, *i.e.* with $J_1 = J_2$.

Setting $M = 1$ in the above equations we should get a state in the un-orbifolded theory, *i.e.* in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. Indeed, it is easily seen that (3.13)–(3.14) becomes equal to (2.2)–(2.3) in the case $\alpha = 1/2$ and $r = 1$. In particular, we see from (2.5) that

¹⁰Here and in the following we are considering the branch of \arctan with $\arctan(0) = 0$.

$n = -2m$. Thus, for $M = 1$ the orbifolded circular string state reduces to the circular string state with $\alpha = 1/2$.

Regarding now the Bethe equations (2.2)–(2.3) for the circular string we observe that setting $\alpha = 1/2$ and $r = 1$ and making the formal replacement $m \rightarrow m/M$ we get the equations for the orbifolded circular string (3.13)–(3.15). Since in solving equations (2.2)–(2.3) one does not use that m is an integer we can thus get the solution for the orbifolded circular string state merely by making the replacement $m \rightarrow m/M$ in the solution (2.32). This reveals that the one-loop energy for the orbifolded circular string state is

$$\mathcal{E} = \frac{\lambda' J^2 m^2}{2M^2(J-1)} - \frac{\pi^2}{6} \frac{\lambda' J^2 m^4}{M^4(J-1)^2} + \mathcal{O}((m/M)^6) \quad (3.16)$$

for $J \gg 1$. We can furthermore infer from the above considerations that the result (2.8) for the circular string one-loop energy to order λ'/J can be extended to give the full one-loop energy for the orbifolded circular string state to order λ'/J

$$\mathcal{E} = \frac{\lambda' J m^2}{2M^2} + \frac{\lambda' m^2}{2M^2} + \frac{\lambda'}{2} \sum_{n=1}^{\infty} \left[n^2 \sqrt{1 - \frac{4m^2}{n^2 M^2}} - n^2 + \frac{2m^2}{M^2} \right] + \mathcal{O}(\lambda' J^{-1}) \quad (3.17)$$

We can now introduce the *winding state* as the orbifolded circular string state in the limit of large J and large M with J/M^2 fixed. To obtain the energy from that of the orbifolded circular string state (3.16) we should consider how this limit affects the derivation of the energy in section 2.1. Due to the replacement $m \rightarrow m/M$ we see that the somewhat formal expansion (2.10) in m that we introduced in section 2.1 is replaced by the expansion

$$f(J, \alpha, m) = \frac{m^2}{M^2} g_1(J, \alpha) + \frac{m^4}{M^4} g_2(J, \alpha) + \frac{m^6}{M^6} g_3(J, \alpha) + \dots \quad (3.18)$$

which, for $M \gg m$, is a perturbative expansion in m/M . Therefore, the steps in the derivation of the energy in section 2.1 remain valid also in the limit $M \rightarrow \infty$. In fact, the regime in which the derivation in section 2.1 is valid is larger for the winding state since in the large M limit the non-perturbative effects are negligible. This is most easily seen by considering (3.17) from which one can see that for $M \rightarrow \infty$ we can ignore completely the non-perturbative contributions to the energy. In detail, from (3.16) we see that the one-loop energy of the winding state is

$$\mathcal{E} = \lambda' \frac{\tilde{m}^2}{2(1-1/J)} - \lambda' \frac{\pi^2 \tilde{m}^4}{6(J-1)^2} + \mathcal{O}((m/M)^6) \quad (3.19)$$

for $J \gg 1$ and where we defined

$$\tilde{m} \equiv \frac{\sqrt{J}}{M} m \quad (3.20)$$

which is fixed in the limit $M, J \rightarrow \infty$. Expanding in $1/J$, we get from (3.19)

$$\mathcal{E} = \frac{\lambda' \tilde{m}^2}{2} \left[1 + \frac{1}{J} + \frac{1}{J^2} + \frac{1}{J^3} - \frac{\pi^2 \tilde{m}^2}{3} \left(\frac{1}{J^2} + \frac{2}{J^3} \right) + \frac{C \tilde{m}^4}{J^3} + \mathcal{O}(J^{-4}) \right] \quad (3.21)$$

where C is a constant which is not determined from (3.19). From this expression it is clear that for each order in $1/J$ we only have a finite number of terms in powers of m . Thus, as advertised in section 2.1, we see that for the winding state we can find the expression for the energy for $m = 1$ by making an expansion in m/M , since for each power of $1/J$ we only have a finite number of terms in the m/M expansion.

Finally, we can combine the result (3.21) with the result for the energy that one gets from (3.17) in the large M expansion. In this way we can get the leading contribution to the energy for each power of m . In particular, for m^6 this is $-2\zeta(4)\lambda'\tilde{m}^6/J^3$ while for m^8 it is $-5\zeta(6)\lambda'\tilde{m}^8/J^4$. Using the m^6 term we determine $C = -4\zeta(4)$ in (3.21), revealing the following one-loop energy for the winding state to order λ'/J^3

$$\mathcal{E} = \frac{\lambda'\tilde{m}^2}{2} \left[1 + \frac{1}{J} + \frac{1}{J^2} + \frac{1}{J^3} - \frac{\pi^2\tilde{m}^2}{3} \left(\frac{1}{J^2} + \frac{2}{J^3} \right) - \frac{2\pi^4\tilde{m}^4}{45J^3} + \mathcal{O}(J^{-4}) \right] \quad (3.22)$$

for $J \gg 1$.

3.2 Winding state from Landau-Lifshitz sigma-model

We now consider the $\mathcal{N} = 2$ orbifolded supersymmetric quiver gauge theory version of the LL model. To obtain the sigma-model that describes it one should use the one-loop Hamiltonian for the $\mathcal{N} = 2$ orbifolded supersymmetric quiver gauge theory acting on operators of the form (3.9). However, since this Hamiltonian corresponds to the one of $XXX_{1/2}$ Heisenberg spin-chain with twisted boundary conditions, the only difference with respect to the $\mathcal{N} = 4$ SYM is that φ is not periodic in 2π but instead it has the periodicity

$$\varphi \equiv \varphi + \frac{4\pi}{M} \quad (3.23)$$

Thus, the sigma model action is still given by eq. (2.35), but with the S^2 target space replaced by an S^2 with the identifications (3.23).

Before studying fluctuations around the winding mode in the orbifolded LL model we first briefly consider the case without winding mode $m = 0$ and without the orbifolding $M = 1$. In this case we quantize the LL action by zooming in on the point $(\theta, \varphi) = (0, 0)$ (corresponding to the point $\vec{n} = (1, 0, 0)$). Thus, we have that the S^2 target-space becomes \mathbb{R}^2 in this limit. As found in [23, 24] a convenient parametrization of S^2 for studying these fluctuations is in the coordinates z_1, z_2 defined in eq. (C.2) in appendix C. This parametrization is particularly useful in that the Hamiltonian only consists of terms with an equal number of annihilation and creation operators. This made the computations at order $1/J^2$ in [23, 24] possible. We therefore use this parametrization of S^2 below to study the fluctuations around the winding mode.

We want to study quantum corrections to the classical energy of the winding state which we denote $|m\rangle$. Classically, a winding mode corresponds to

$$\theta = 0, \quad \varphi = \frac{2m}{M}\sigma \quad (3.24)$$

We want to consider quantum fluctuations around this state using the parametrization (C.2) of the two-sphere. This can be done using the relation eq. (C.3) between the

variables (θ, φ) and the (z_1, z_2) coordinates for the two-sphere. We therefore write

$$\varphi = \frac{2m}{M}\sigma + \arcsin\left(\frac{2f}{\sqrt{J}}\sqrt{\frac{1 - \frac{f^2+g^2}{J}}{1 - \frac{4g^2}{J}\left(1 - \frac{f^2+g^2}{J}\right)}}\right) \quad (3.25)$$

$$\sin\theta = \frac{2g}{\sqrt{J}}\sqrt{1 - \frac{f^2+g^2}{J}} \quad (3.26)$$

This is found from (C.3) by substituting $z_1 = f/\sqrt{J}$ and $z_2 = g/\sqrt{J}$ and by adding the winding to the φ coordinate.

In the following we consider the limit $J, M \rightarrow \infty$ with J/M^2 fixed. In this limit we get that the target space, which is a two-sphere with identifications (3.23), becomes a cylinder $S^1 \times \mathbb{R}$ with f parameterizing the circle of radius \sqrt{J}/M and g parameterizing \mathbb{R} .

The Lagrangian can be written in a $1/J$ expansion as

$$J\mathcal{L} = 4gf\dot{\sigma} - \mathcal{H}_2 - \frac{1}{J}\mathcal{H}_4 - \frac{1}{J^2}\mathcal{H}_6 + \dots \quad (3.27)$$

with

$$\mathcal{H}_2 = (f' + \tilde{m})^2 + (g')^2 \quad (3.28)$$

where \tilde{m} is defined in (3.20). We see that the leading order Hamiltonian (3.28) correctly corresponds to having a winding mode with winding number m around the circle direction of the cylinder $S^1 \times \mathbb{R}$ with radius \sqrt{J}/M . The linearized EOMs are

$$\dot{f} = -\frac{1}{2}g'', \quad \dot{g} = \frac{1}{2}f'' \quad (3.29)$$

Their solution can be written in the following way

$$f = \frac{1}{2}(A + A^\dagger), \quad g = \frac{i}{2}(A^\dagger - A) \quad (3.30)$$

where we introduced

$$A = \sum_{n \neq 0} a_n e^{-i\frac{n^2}{2}t + in\sigma} \quad (3.31)$$

Turning to the $1/J$ and $1/J^2$ corrections in the Lagrangian (3.27), we find from a classical computation the following terms

$$\mathcal{H}_4 = 2(ff' + gg')^2 - (f^2 + g^2)((f')^2 + (g')^2) + (f'f^2 - 5f'g^2 + 6fgg')\tilde{m} - 4\tilde{m}^2g^2 \quad (3.32)$$

$$\begin{aligned} \mathcal{H}_6 = & (f^2 + g^2)(ff' + gg')^2 + \left(g(f^2 + g^2)(gf' - f'g) + \frac{3}{4}(f^2 + g^2)^2 f'\right)\tilde{m} \\ & + 2\tilde{m}^2(2g^4 + (fg)^2 + (gf)^2) \end{aligned} \quad (3.33)$$

up to total derivatives with respect to the time t . Note that the m -independent terms match those found in [23, 24]. As usual, the Hamiltonian is

$$H = \frac{\lambda'}{4\pi} \int_0^{2\pi} d\sigma \left(\mathcal{H}_2 + \frac{1}{J}\mathcal{H}_4 + \frac{1}{J^2}\mathcal{H}_6 + \dots \right) \quad (3.34)$$

Now we can treat these Hamiltonians as perturbations of the free Hamiltonian (3.28) and compute the energy of the winding state perturbatively.

In the following we want to use the parts of (3.32)–(3.33) proportional to m^2 to compute the corrections to the energy of the winding state. As part of this, we need to consider the vacuum expectation value of \mathcal{H}_4 and \mathcal{H}_6 . As we discuss below, this is trivially zero for the m -independent terms and the terms proportional to m since they contain derivatives with respect to σ . However, for the terms proportional to m^2 the expectation value can depend on the ordering of the f and g variables. Their commutation relations are given in eq. (2.53). In particular, the absence of the zero mode in (3.30), which gives rise to the $-i/2$ term in eq. (2.53), plays a crucial role in our calculation.¹¹ The quantum operators corresponding to (3.25) and (3.26) can be consistently defined in an expansion for large J . Using ζ -function regularization one can show that, even at the same point σ , f and g do not commute and they have the following commutation relation

$$[f(t, \sigma), g(t, \sigma)] = -\frac{i}{2} \tag{3.35}$$

Consequently f and g can be treated as non-commuting coordinates of a 2 dimensional space and one can use the general procedure of non-commutative geometry, *i.e.* the Weyl prescription, for associating a quantum operator to a classical function [40]. This technique provides a systematic way to describe noncommutative spaces in general. Although we will focus solely on the commutator (3.35), Weyl quantization also works for more general commutation relations.

Let us consider the commutative algebra of functions on a 2 dimensional Euclidean space \mathbb{R}^2 , with product defined by the usual multiplication of functions. In general a function $F(f, g)$ may be described by its Fourier transform

$$\tilde{F}(k_f, k_g) = \int df dg e^{-ik_f f - ik_g g} F(f, g) \tag{3.36}$$

with $\tilde{F}(-k) = \tilde{F}(k)$ whenever $F(x)$ is real-valued. We define a noncommutative space by replacing the local coordinates f and g of \mathbb{R}^2 by Hermitian operators \hat{f} and \hat{g} obeying the commutation relations (3.35). The \hat{f} and \hat{g} then generate a noncommutative algebra of operators.

Given the function $F(f, g)$ and its corresponding Fourier coefficients (3.36), we introduce its Weyl symbol by

$$W[F] = \int \frac{dk_f dk_g}{(2\pi)^2} \tilde{F}(k) e^{ik_f \hat{f} + ik_g \hat{g}}, \tag{3.37}$$

where we have chosen the symmetric Weyl operator ordering prescription. The Weyl operator $\hat{W}[F]$ is Hermitian if $F(f, g)$ is real-valued. We can write (3.37) in terms of an explicit map $\hat{M}(\hat{f}, \hat{g})$ between operators and fields by using (3.36) to get

$$\hat{W}[F] = \int df dg F(f, g) \hat{M}(\hat{f}, \hat{g}) \tag{3.38}$$

¹¹There is no zero-mode since it cannot correspond to a physical excitation. The absence of the zero-mode means that we are instead in a sector of fixed total spin $S_z = -i\partial_\varphi = (J_1 - J_2)/2$. See also [34, 39].

where

$$\hat{M}(\hat{f}, \hat{g}) = \int \frac{dk_f dk_g}{(2\pi)^2} e^{ik_f \hat{f} + ik_g \hat{g}} e^{-ik_f f - ik_g g} \quad (3.39)$$

The operator (3.39) is Hermitian, $\hat{M}^\dagger = \hat{M}$, and it describes a mixed basis for operators and fields on spacetime. In this way we may interpret the field $F(f, g)$ as the coordinate space representation of the Weyl operator $\hat{W}[F]$. Note that in the commutative case, the map (3.39) reduces trivially to a delta-function. But generally, by the Baker-Campbell-Hausdorff (BCH) formula, it is a highly non-trivial field operator. Using (3.35) the BCH formula gives

$$e^{ik_f \hat{f} + ik_g \hat{g}} = e^{ik_f \hat{f}} e^{ik_g \hat{g}} e^{-\frac{i}{4} k_f k_g} \quad (3.40)$$

When this is inserted in (3.37) one can construct explicitly the quantum operator corresponding to any classical polynomial in g and f . Consider for example eq. (3.26), expanding for large J we get

$$\sin \theta = \frac{2g}{\sqrt{J}} \left[1 - \frac{f^2 + g^2}{J} - \frac{(f^2 + g^2)^2}{4J^2} + \mathcal{O}(J^{-3}) \right] \quad (3.41)$$

Using the definitions above the corresponding Weyl ordered operator (3.38) reads

$$\begin{aligned} W[\sin \theta] &= \frac{2\hat{g}}{\sqrt{J}} - \frac{1}{J^{3/2}} \left[(\hat{f}^2 + \hat{g}^2)\hat{g} + \frac{i}{2}\hat{f} \right] \\ &+ \frac{i}{4J^{5/2}} \left[i(\hat{f}^2 + \hat{g}^2)^2 \hat{g} - \hat{f}(\hat{f}^2 + \hat{g}^2) - \frac{i}{4}\hat{g} \right] + \mathcal{O}(J^{-7/2}) \end{aligned} \quad (3.42)$$

We can therefore construct explicitly the Hamiltonian densities appearing in (3.27) which are relevant at each order in the $1/J$ expansion. The terms in the Hamiltonian proportional to m^2 are $\tilde{m}^2 \cos^2 \theta$ which thus become

$$\tilde{m}^2(1 - W[\sin \theta]^2) = \tilde{m}^2 \left[1 - \frac{4}{J}\hat{g}^2 + \frac{2}{J^2}(2\hat{g}^4 + (\hat{f}\hat{g})^2 + (\hat{g}\hat{f})^2) + \mathcal{O}(J^{-3}) \right] \quad (3.43)$$

Upon removing the hats we get precisely the m^2 parts in (3.32) and (3.33).

Since the winding state $|m\rangle$ is our vacuum state we require it to satisfy

$$A|m\rangle = 0 \quad (3.44)$$

From the commutator (3.35) we see that A and A^\dagger have the commutator (at the same point)

$$[A, A^\dagger] = -1 \quad (3.45)$$

Using this, we obtain that the vacuum state $|m\rangle$ satisfies the following properties

$$\langle m|AA^\dagger|m\rangle = -1, \quad \langle m|(AA^\dagger)^2|m\rangle = 1, \quad \langle m|A^2(A^\dagger)^2|m\rangle = 2 \quad (3.46)$$

In addition, we have that any vacuum expectation value involving A' and A'^\dagger is zero.

The Hamiltonian densities can then be written as

$$\mathcal{H}_2 = \frac{1}{2}(A'A'^{\dagger} + A'^{\dagger}A') + (A' + A'^{\dagger})\tilde{m} + \tilde{m}^2 \quad (3.47)$$

$$\mathcal{H}_4 = (\dots) + (\dots)\tilde{m} + (A^2 + (A^{\dagger})^2 - AA^{\dagger} - A^{\dagger}A)\tilde{m}^2 \quad (3.48)$$

$$\mathcal{H}_6 = (\dots) + (\dots)\tilde{m} + \left(\dots + \frac{1}{2}A^2(A^{\dagger})^2\right)\tilde{m}^2 \quad (3.49)$$

where in \mathcal{H}_4 and in \mathcal{H}_6 we only wrote the relevant terms that can contribute to the computation of the energy corrections of the winding state.

The classical value of the energy of the winding state is given by

$$\mathcal{E}_0 = \langle m|H_2|m\rangle = \frac{\lambda'\tilde{m}^2}{2} \quad (3.50)$$

The $1/J$ correction to the classical energy is

$$\mathcal{E}_1 = \langle m|H_4|m\rangle = \frac{\lambda'\tilde{m}^2}{2J} \quad (3.51)$$

We then go on to compute the $1/J^2$ correction. This is given by the second order perturbation theory formula

$$\mathcal{E}_2 = \sum_{|i\rangle \neq |m\rangle} \frac{|\langle m|H_4|i\rangle|^2}{\mathcal{E}_m - \mathcal{E}_i} + \langle m|H_6|m\rangle = \mathcal{E}_A + \mathcal{E}_B \quad (3.52)$$

where $|i\rangle$ is an intermediate state. The computation of \mathcal{E}_B is immediately done and we get

$$\mathcal{E}_B = \langle m|H_6|m\rangle = \frac{\lambda'\tilde{m}^2}{2J^2} \quad (3.53)$$

We now compute \mathcal{E}_A . It is not difficult to see that the only possibility for obtaining a non-zero result is that the intermediate state $|i\rangle$ is of the form

$$|i\rangle \equiv a_n^{\dagger}a_{-n}^{\dagger}|m\rangle \quad (3.54)$$

We compute $[[A^2|_{t=0}, a_n^{\dagger}], a_{-n}^{\dagger}] = 2$ and from this we get $[[\mathcal{H}_4, a_n^{\dagger}], a_{-n}^{\dagger}] = 2\tilde{m}^2$. Thus we have

$$\langle m|\mathcal{H}_4|i\rangle = \langle m|\mathcal{H}_4a_n^{\dagger}a_{-n}^{\dagger}|m\rangle = \langle m|[[\mathcal{H}_4, a_n^{\dagger}], a_{-n}^{\dagger}]|m\rangle = 2\tilde{m}^2 \quad (3.55)$$

where $n \neq 0$. Moreover we have that $\mathcal{E}_m = \tilde{m}^2$ and that the intermediate state $|i\rangle$ has energy $\mathcal{E}_i = \tilde{m}^2 + 2n^2$. Therefore we have

$$\mathcal{E}_A = \sum_{n=1}^{\infty} \frac{|\langle m|H_4|i\rangle|^2}{\mathcal{E}_m - \mathcal{E}_i} = -\frac{4\lambda'\tilde{m}^4}{J^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{2\lambda'\tilde{m}^4\pi^2}{3J^2} \quad (3.56)$$

Up to this order we can write the corrected energy of the winding state as

$$\mathcal{E} = \frac{\lambda'}{2}\tilde{m}^2 \left[1 + \frac{1}{J} + \frac{1}{J^2} \left(1 - \frac{\pi^2}{3}\tilde{m}^2 \right) + \mathcal{O}(J^{-3}) \right] \quad (3.57)$$

We note that in the orbifold case, contrary to the usual case, there is no higher derivative term that has to be included up to this order. We see that eq. (3.57) precisely matches the result obtained from the Bethe ansatz (3.22). If we set $M = 1$, $r = 1$ and $\alpha = 1/2$, it also matches the Bethe ansatz result (2.33).

It is interesting to test whether our results above relies on choosing the Weyl ordering for f and g in (3.42). More generally, we can take the classical expression (3.41) and try the most general way to multiply g with $f^2 + g^2$ while retaining the same classical limit. This is parameterized as the following more general ordering prescription for (3.41)

$$W_{\text{gen}}[\sin \theta] = \frac{2\hat{g}}{\sqrt{J}} - \frac{(1 - a_1)\hat{g}(\hat{f}^2 + \hat{g}^2) + a_1(\hat{f}^2 + \hat{g}^2)\hat{g}}{J^{3/2}} - \frac{(1 - a_2 - a_3)\hat{g}(\hat{f}^2 + \hat{g}^2)^2 + a_2(\hat{f}^2 + \hat{g}^2)\hat{g}(\hat{f}^2 + \hat{g}^2) + a_3(\hat{f}^2 + \hat{g}^2)^2\hat{g}}{4J^{5/2}} + \mathcal{O}(J^{-7/2}) \quad (3.58)$$

The Weyl ordering corresponds to the particular choice $a_1 = 1/2$, $a_2 = -1/2$ and $a_3 = 3/4$. Clearly, using this ordering cannot affect the $-4\tilde{m}^2\hat{g}^2/J$ term at order $1/J$ which gives rise to the m^4/J^2 correction to the energy of the winding state. However, for the m^2/J^2 correction, which comes from the expectation value $\langle m|H_6|m\rangle$, it could have an effect. Using (3.58) we compute

$$\tilde{m}^2(1 - W_{\text{gen}}[\sin \theta]^2) = \tilde{m}^2 \left[1 - \frac{4}{J}\hat{g}^2 + \frac{2}{J^2} \left[2\hat{g}^4 + 2(\hat{f}\hat{g})^2 + ia_1(\hat{f}\hat{g} + \hat{g}\hat{f}) \right] + \mathcal{O}(J^{-3}) \right] \quad (3.59)$$

Computing now the expectation value $\langle m|H_6|m\rangle$ we find that it is equal to (3.53) for any value of a_1 , a_2 and a_3 . Thus, amazingly, the expectation value $\langle m|H_6|m\rangle$ is independent of the choice of ordering prescription. This makes it a rather solid prediction.

Higher orders. Going to the next order, namely λ'/J^3 , is a non-trivial task and one has first to construct the quantum Hamiltonian \mathcal{H}_8 using the Weyl prescription and then use perturbation theory to the next order. The quantum expression for \mathcal{H}_8 and the perturbative calculations are too complicated to display here, but for the term proportional to \tilde{m}^2 they give¹²

$$\frac{\lambda'}{8J^3}\tilde{m}^2 \quad (3.60)$$

This, as expected, is not in agreement with the Bethe ansatz result (3.22). To get to (3.60) one has to use ζ -function regularization to define the sums over intermediate states in perturbation theory and this regularization, as was argued in ref. [23], is not expected to give the correct results at this order. The same arguments apply for the term proportional to \tilde{m}^4 where, moreover, in the sigma model action (2.35) one should include also the higher derivative term in (2.41). More precisely, we should add the following term to the Hamiltonian

$$H_{h.d.} = \frac{J\lambda'}{4\pi} \int_0^{2\pi} d\sigma \left[-\frac{\pi^2}{12J^2}(\tilde{n}'')^2 + \mathcal{O}(J^{-4}) \right] \quad (3.61)$$

¹²Note here that one can find values of a_1 , a_2 and a_3 such that the expectation value of the $1/J^3$ correction to the Hamiltonian gives $\frac{\lambda'}{2J^3}\tilde{m}^2$ which is the result predicted by the gauge theory (3.22). However this does not seem the correct way to resolve the discrepancy.

Evaluated on the winding state this Hamiltonian gives

$$\langle m|H_{h.d.}|m\rangle = -\frac{2\lambda'\pi^2}{3J^3}\tilde{m}^4 \tag{3.62}$$

Putting all together we do not get a result which is consistent with the solution of the Bethe equations. It is however possible to provide an ordering prescription, which differs from the Weyl ordering only starting from the order λ'/J^3 , and that gives instead results for the λ'/J^3 terms that match precisely those of the Bethe equations.

4. Conclusions

The main goal of this paper has been to investigate how well the continuous Landau-Lifshitz sigma model reproduces the results obtained from the Bethe equations that provide the one-loop scaling dimensions of operators in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. This has been examined by computing the finite size corrections to the energy of the circular rotating string and to its orbifolded generalization. In particular we also considered the winding state which is obtained as the limit of the orbifolded circular string solution when $J \rightarrow \infty$ and J/M^2 is kept fixed.

Our conclusion is that, in the case of the circular rotating string, for the leading $1/J$ corrections in m^2 and m^4 and for the subleading $1/J^2$ corrections to the m^2 term, we found, in fact, complete agreement between the perturbative LL energies and the solutions of the Bethe equations. In the case of the winding state we obtain full agreement of the finite-size corrections up to, and including, the $1/J^2$ order. In particular, the matching of the m^4/J^2 term represents an important example of a successful match of gauge theory and a continuous sigma-model which relies on summing over the so called “impurity non preserving channels”, namely intermediate states excited by a number of oscillators different from the external state.

The Bethe equations have been solved by using a novel procedure which allows to determine the finite size corrections to the dimensions of these operators and the energies of the corresponding string states in an expansion in the winding number m but to all orders in the angular momentum J . We also applied the same procedure to solve the Bethe equations for the $XXX_{-1/2}$ Heisenberg spin chain, which instead describe the one-loop contribution to the $\mathfrak{sl}(2)$ sector. According to the AFS ansatz [12] the solution in this case should describe, on the string side of the AdS/CFT duality, the spinning string.

We were then able to match the results of the $\mathfrak{su}(2)$ sectors with those derived from an effective quantum LL model, both in its orbifolded and un-orbifolded versions. The matching is highly non-trivial since it requires in the LL model second order perturbation theory and ζ -function regularization.

The calculations of quantum finite-size corrections are much simpler in the LL model than in the full superstring computation since in this model one does not include the contributions of the bosonic and fermionic modes which do not belong to the $\mathfrak{su}(2)$ sector. Omitting these and other string modes is obviously not correct in general, but in some simple cases it may happen that the role of these extra modes may be just to provide a

particular UV regularization of the quantum LL result. This was first suggested in [28] and it is by now established that the ζ -function regularization is the correct one for reproducing the first order finite size correction. Our results for the circular rotating string and for the winding state confirm this observation. However the ζ -function regularization might not be the appropriate one when comparing the results of the next to leading order finite size corrections, as suggested also in [23, 24]. The mismatch that we found for the m^4/J^2 term in the case of the circular rotating string, and for the m^2/J^3 term in the case of the winding state, clearly point towards the necessity of a full superstring calculation that would naturally suggest the correct regularization prescription and hopefully provide complete agreement with the Bethe ansatz results.

In a subsequent paper [39] we shall show that, for the pp-wave background studied in [34], the superstring degrees of freedom which are relevant at the first order in λ' are precisely those of the $\mathfrak{su}(2)$ sector and, consequently, the superstring sigma model reduces to the LL model in the particular set of coordinates used here to study the winding state. In this context we shall also compute directly from the superstring theory the $1/J^2$ corrections to the string excitations. For the winding state we shall show that for the m^2/J term all the divergences in the LL model, which we need to regularize, cancel in the superstring calculation when including the zero-mode contributions from the full superstring theory, including fermions and all the transverse directions, and the finite piece that remains is precisely (3.51). This strongly suggests that the full superstring theory picks up a particular regularization prescription which allows to match the gauge theory side.

There are some interesting extensions of this work. An obvious generalization is the analysis of the $\mathfrak{sl}(2)$ sector of operators of the type $\text{Tr}(D^{s_1} Z D^{s_2} Z \dots D^{s_J} Z)$. The spin chain LL action also in this case should match the string action [43, 44] and consequently a matching of the higher order finite size corrections between the solution of the Bethe equations (1.2) and the LL results should be easily obtained along the lines of the present paper.

It would be interesting to study also the β -deformed version of AdS/CFT which relates an exactly marginal superconformal deformation of SYM theory to string theory in the $AdS_5 \times (S^5)_\beta$ background constructed by applying a combination of T -duality, shift of angle and another T -duality to the original $AdS_5 \times S^5$ background [57]. The existence of integrable structures on the two sides of the duality was first discussed in [58] where it was argued that the integrability of strings in $AdS_5 \times S^5$ implies the integrability of the deformed world sheet theory with real deformation parameter. The Bethe equations are identical to those solved here for the orbifold case. The β -deformed Bethe equations in fact contain a twist as in the orbifold but with $2m/M \rightarrow \beta J$. Also in this case one can focus on the solutions of the Bethe equations with equal mode numbers for all the roots and compute the higher order finite size corrections with the procedure developed in this paper. The results can then be compared with one loop string theory results and with possible LL calculations [58]. This might lead to a better understanding of the spectrum of strings in less-supersymmetric backgrounds.

Finally, it would be important to study if the procedure used in this paper to solve the Bethe equations could be extended also to higher powers of the winding number of the string states.

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A. Zeroes of the Laguerre polynomial

The Laguerre differential equation is

$$\left[y \frac{d^2}{dy^2} + (\nu + 1 - y) \frac{d}{dy} + \mu \right] f(y) = 0 \quad (\text{A.1})$$

The associated Laguerre polynomial, which is a solution of (A.1), is written $f = L_\mu^\nu(y)$.

We write the zeroes of the Laguerre polynomial as y_k , $k = 1, 2, \dots, \mu$, where we here used the general property of the Laguerre polynomial that the number of zeroes equals μ . For the zeroes of the Laguerre polynomial we have the sum rules [30]

$$y_k \sum_{j=1, j \neq k}^{\mu} (y_k - y_j)^{-1} = -\frac{1}{2} [1 + \nu - y_k] \quad (\text{A.2})$$

$$y_k^3 \sum_{j=1, j \neq k}^{\mu} (y_k - y_j)^{-3} = -\frac{1}{8} [(\nu + 1)(\nu + 3) - (2\mu + \nu + 1)y_k] \quad (\text{A.3})$$

B. Circular rotating string

In this appendix we repeat the computation of section 2.2 using a different coordinate system.

We consider again fluctuations around

$$\theta = 0, \quad \varphi = 2m\sigma \quad (\text{B.1})$$

for which the classical energy is

$$E_0 = J \left(1 + \lambda' \frac{m^2}{2} + \mathcal{O}(\lambda'^2) \right) \quad (\text{B.2})$$

We want to compute quantum corrections to the energy of the ground state $|0_m\rangle$. In section 2.2 we used the parametrization (C.1) for the two-sphere in doing this. Here we instead choose to consider the Landau-Lifshitz action using the parametrization (C.2). Using (C.3) we see that this means we should consider the following form for the fluctuations around the solution (B.1)

$$\varphi = 2m\sigma + \arcsin \left(\frac{2f}{\sqrt{J}} \sqrt{\frac{1 - \frac{f^2+g^2}{J}}{1 - \frac{4g^2}{J} \left(1 - \frac{f^2+g^2}{J} \right)}} \right) \quad (\text{B.3})$$

$$\sin \theta = \frac{2g}{\sqrt{J}} \sqrt{1 - \frac{f^2+g^2}{J}} \quad (\text{B.4})$$

where m is the winding number. The Lagrangian is given again by eq. (2.45) but now we have the following identification

$$\mathcal{H}_2 = (f')^2 + (g')^2 - 4m^2g^2 \quad (\text{B.5})$$

$$\mathcal{H}_3 = (f'f^2 - 5f'g^2 + 6fgg')m \quad (\text{B.6})$$

$$\mathcal{H}_4 = 2(ff' + gg')^2 - (f^2 + g^2)[(f')^2 + (g')^2] - 4m^2[g^4 + (fg)^2] \quad (\text{B.7})$$

The solution to the linearized EOMs is still given by eq. (2.51) with the same definition (2.52) for ω_n and w_n . As we can see, the quadratic Hamiltonian H_2 is the same in the two parametrization, as expected. Therefore, for the leading correction to the energy of the ground state, we get the same expression (2.57), after regularizing appropriately the divergences.

We now move to the next subleading correction to the energy which is given by a second order perturbation theory expression as in section 2.2, namely

$$E_2 = \sum_{|i\rangle \neq |0_m\rangle} \frac{|\langle 0_m | H_3 | i \rangle|^2}{E_{0_m} - E_i} + \langle 0_m | H_4 | 0_m \rangle = E_2^1 + E_2^2 \quad (\text{B.8})$$

where $|i\rangle$ is an intermediate state. Even though the expression of H_3 is very different in the two parametrizations, it is not difficult to see that both give the same result for $\langle 0_m | H_3 | i \rangle$. This means that $E_2^{(1)}$ is again given by eq. (2.61).

The evaluation of $E_2^{(2)}$ is instead slightly different in this case, therefore we present it in detail. We have

$$\begin{aligned} & \frac{\lambda'}{4\pi J} \int_0^{2\pi} d\sigma [2(ff' + gg')^2 - (f^2 + g^2)((f')^2 + (g')^2)] \\ &= \frac{\lambda'}{2J} \sum_{l,p,q,s=-\infty}^{\infty} \frac{\sqrt{w_l w_p w_q w_s}}{16} \delta_{l+p+q+s} (s-2p)q \\ & \times \left[\left(1 - \frac{1}{w_l w_p}\right) \left(1 - \frac{1}{w_q w_s}\right) (a_l a_p a_{-q}^\dagger a_{-s}^\dagger + a_{-l}^\dagger a_{-p}^\dagger a_q a_s) \right. \\ & \left. + \left(1 + \frac{1}{w_l w_p}\right) \left(1 + \frac{1}{w_q w_s}\right) (a_{-l}^\dagger a_p + a_l a_{-p}^\dagger) (a_{-s}^\dagger a_q + a_s a_{-q}^\dagger) \right] \quad (\text{B.9}) \end{aligned}$$

and

$$\begin{aligned} & \frac{\lambda'}{4\pi J} \int_0^{2\pi} d\sigma 4m^2 [g^4 + (fg)^2] = 2\frac{m^2}{J} \lambda' \sum_{l,p,q,s=-\infty}^{\infty} \frac{\delta_{l+p+q+s}}{16\sqrt{w_l w_p w_q w_s}} \\ & \times \left[(1 + w_l w_q) (a_l a_p a_{-q}^\dagger a_{-s}^\dagger + a_{-l}^\dagger a_{-p}^\dagger a_q a_s + a_l a_{-p}^\dagger a_{-q}^\dagger a_s \right. \\ & \left. + a_{-l}^\dagger a_p a_q a_{-s}^\dagger) + (1 - w_l w_q) (a_l a_{-p}^\dagger a_q a_{-s}^\dagger + a_{-l}^\dagger a_p a_{-q}^\dagger a_s) \right] \quad (\text{B.10}) \end{aligned}$$

where we only kept terms with two a 's and two a^\dagger 's. We get

$$\begin{aligned} \langle E_2^2 = 0_m | H_4 | 0_m \rangle &= \frac{\lambda'}{2J} \sum_{l,p=-\infty}^{\infty} \left\{ \frac{1}{32} \left\{ \left[(p^2 + 6pl) \left(w_l w_p + \frac{1}{w_l w_p} \right) \right. \right. \right. \\ & \left. \left. \left. - (p^2 - 2pl) \left(\frac{w_l}{w_p} + \frac{w_p}{w_l} \right) - 4(p^2 + 2pl) \right] + \frac{m^2}{4} \frac{(3 + w_l^2)}{w_l w_p} \right\} \right\} \quad (\text{B.11}) \end{aligned}$$

In the previous expression there are divergent sums, some of which can be regularized as before by adding and subtracting the divergent contribution and then using ζ -function regularization. We get

$$\begin{aligned}
 E_2^2 = \frac{\lambda'}{2J} & \left\{ \frac{1}{16} \sum_{l=-\infty}^{\infty} \sqrt{1 - \frac{4m^2}{l^2}} \sum_{p=1}^{\infty} \left[\left(2p\sqrt{p^2 - 4m^2} - \frac{p^2}{\sqrt{1 - \frac{4m^2}{p^2}}} \right) + 4m^2 \right] \right. \\
 & - \frac{1}{16} \left[2 \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt{1 - \frac{4m^2}{l^2}}} - 1 \right) - 1 \right] \sum_{p=1}^{\infty} \left[\left(2p\sqrt{p^2 - 4m^2} - \frac{p^2}{\sqrt{1 - \frac{4m^2}{p^2}}} \right) \right. \\
 & \left. \left. + 4m^2 \right] + \frac{3m^2}{4} \left[2 \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt{1 - \frac{4m^2}{l^2}}} - 1 \right) - 1 \right] \left[2 \sum_{p=1}^{\infty} \left(\frac{1}{\sqrt{1 - \frac{4m^2}{p^2}}} - 1 \right) - 1 \right] \right. \\
 & \left. + \frac{m^2}{4} \sum_{l=-\infty}^{\infty} \sqrt{1 - \frac{4m^2}{l^2}} \left[2 \sum_{p=1}^{\infty} \left(\frac{1}{\sqrt{1 - \frac{4m^2}{p^2}}} - 1 \right) - 1 \right] \right\} \quad (B.12)
 \end{aligned}$$

Here we see that the sum over l in the first and last term on the r.h.s. of the previous expression is divergent. To obtain a meaningful result, those divergences need to be removed. We will show that for the terms proportional to m^2 and m^4 those divergences exactly cancel and we conjecture that the same should happen for all the other terms in the series expansion in powers of m^2 . Expanding the result obtained up to this order for the correction to the energy of the circular string in powers of m^2 , up to m^4 , we obtain

$$E - J = J\lambda' \left[\frac{m^2}{2} + \frac{1}{J} \left(\frac{m^2}{2} - \frac{\pi^2}{6} m^4 + \mathcal{O}(m^6) \right) + \frac{1}{J^2} \left(\frac{m^2}{2} - \frac{4\pi^2}{3} m^4 + \mathcal{O}(m^6) \right) \right] \quad (B.13)$$

where we also included the contribution from the higher derivative term (2.43). We see that (B.13) coincides with (2.68) obtained in section 2.2. It is important to notice that in (B.13) all the divergent contributions present in (B.12) which cannot be regularized, exactly cancel up to the order m^4 .

C. Parameterizations of the two-sphere

We use in this paper two different parameterizations of the two-sphere. The three-dimensional vector $\vec{n} = (n_1, n_2, n_3)$ defines the unit two-sphere as $\vec{n}^2 = 1$. The unit vector \vec{n} can be parameterized by the spherical coordinates θ and φ as

$$n_1 = \cos \theta \cos \varphi, \quad n_2 = \cos \theta \sin \varphi, \quad n_3 = \sin \theta \quad (C.1)$$

Here $-\pi/2 \leq \theta \leq \pi/2$. Note that in this parametrization the equator is at $\theta = 0$, which corresponds to $n_3 = 0$. In particular $(\theta, \varphi) = (0, 0)$ corresponds to $\vec{n} = (1, 0, 0)$. Another parametrization that we use is chosen such that it is symmetric in exchanging n_2 and n_3 . It was found in [23]. It parameterizes \vec{n} by the two coordinates z_1 and z_2 as

$$n_1 = \sqrt{1 - 4z_1^2(1 - z_2^2)}, \quad n_2 = 2z_1\sqrt{1 - z_2^2}, \quad n_3 = 2z_2\sqrt{1 - z_2^2} \quad (C.2)$$

where z is $z = \sqrt{z_1^2 + z_2^2}$.

We see that in this parametrization $(z_1, z_2) = (0, 0)$ corresponds to $\vec{n} = (1, 0, 0)$. The two above parameterizations (θ, φ) and (z_1, z_2) are connected through the following relations

$$\varphi = \arcsin \left(2z_1 \sqrt{\frac{1 - z^2}{1 - 4z_2^2(1 - z^2)}} \right), \quad \sin \theta = 2z_2 \sqrt{1 - z^2} \quad (\text{C.3})$$

References

- [1] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [[hep-th/0204051](#)].
- [2] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)].
- [3] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$* , *Nucl. Phys.* **B 668** (2003) 77 [[hep-th/0304255](#)].
- [4] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)].
- [5] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [[hep-th/0303060](#)].
- [6] N. Beisert and M. Staudacher, *The $\mathcal{N} = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [[hep-th/0307042](#)].
- [7] N. Beisert, *The $\mathfrak{su}(2|3)$ dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [[hep-th/0310252](#)].
- [8] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5* , *Phys. Lett.* **B 543** (2002) 81 [[hep-th/0206103](#)].
- [9] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [[hep-th/0305116](#)].
- [10] D. Serban and M. Staudacher, *Planar $\mathcal{N} = 4$ gauge theory and the Inozemtsev long range spin chain*, *JHEP* **06** (2004) 001 [[hep-th/0401057](#)].
- [11] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [[hep-th/0307032](#)].
- [12] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)].
- [13] N. Beisert, *The $\mathfrak{su}(2|2)$ dynamic S-matrix*, [hep-th/0511082](#).
- [14] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [[hep-th/0610251](#)].
- [15] T. Harmark, K.R. Kristjansson and M. Orselli, *Matching gauge theory and string theory in a decoupling limit of AdS/CFT* , [arXiv:0806.3370](#).
- [16] T. Harmark and M. Orselli, *Quantum mechanical sectors in thermal $N = 4$ super Yang-Mills on $R \times S^3$* , *Nucl. Phys.* **B 757** (2006) 117 [[hep-th/0605234](#)].
- [17] T. Harmark and M. Orselli, *Matching the Hagedorn temperature in AdS/CFT* , *Phys. Rev.* **D 74** (2006) 126009 [[hep-th/0608115](#)].

- [18] T. Harmark, K.R. Kristjansson and M. Orselli, *Magnetic Heisenberg-chain / pp-wave correspondence*, *JHEP* **02** (2007) 085 [[hep-th/0611242](#)].
- [19] T. Harmark, K.R. Kristjansson and M. Orselli, *Decoupling limits of $\mathcal{N} = 4$ super Yang-Mills on $R \times S^3$* , *JHEP* **09** (2007) 115 [[arXiv:0707.1621](#)].
- [20] T. Harmark, K.R. Kristjansson and M. Orselli, *The Hagedorn temperature in a decoupled sector of AdS/CFT*, *Fortschr. Phys.* **55** (2007) 754 [[hep-th/0701088](#)].
- [21] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [[hep-th/0306139](#)].
- [22] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys. B* **671** (2003) 3 [[hep-th/0307191](#)].
- [23] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *$1/J$ corrections to semiclassical AdS/CFT states from quantum Landau-Lifshitz model*, *Nucl. Phys. B* **735** (2006) 127 [[hep-th/0509071](#)].
- [24] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *$1/J^2$ corrections to BMN energies from the quantum long range Landau-Lifshitz model*, *JHEP* **11** (2005) 031 [[hep-th/0510080](#)].
- [25] See for example chapter 5 in E. Fradkin, *Field theories of condensed matter systems*, Addison-Wesley Publishing Company, Redwood City U.S.A. (1991).
- [26] M. Kruczenski, *Spin chains and string theory*, *Phys. Rev. Lett.* **93** (2004) 161602 [[hep-th/0311203](#)].
- [27] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, *Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*, *Nucl. Phys. B* **692** (2004) 3 [[hep-th/0403120](#)].
- [28] N. Beisert, A.A. Tseytlin and K. Zarembo, *Matching quantum strings to quantum spins: one-loop vs. finite-size corrections*, *Nucl. Phys. B* **715** (2005) 190 [[hep-th/0502173](#)].
- [29] R. Hernandez, E. Lopez, A. Perianez and G. Sierra, *Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings*, *JHEP* **06** (2005) 011 [[hep-th/0502188](#)].
- [30] S. Ahmed, M. Bruschi, F. Calogero, M.A. Olshanetsky and A.M. Perelomov, *Properties of the zeros of the classical polynomials and of Bessel functions*, ROME-110-1978.
- [31] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev. D* **69** (2004) 086009 [[hep-th/0311004](#)].
- [32] N. Gromov and V. Kazakov, *Double scaling and finite size corrections in $\mathfrak{sl}(2)$ spin chain*, *Nucl. Phys. B* **736** (2006) 199 [[hep-th/0510194](#)].
- [33] N. Gromov and P. Vieira, *Complete 1-loop test of AdS/CFT*, *JHEP* **04** (2008) 046 [[arXiv:0709.3487](#)].
- [34] M. Bertolini, J. de Boer, T. Harmark, E. Imeroni and N.A. Obers, *Gauge theory description of compactified pp-waves*, *JHEP* **01** (2003) 016 [[hep-th/0209201](#)].
- [35] K. Ideguchi, *Semiclassical strings on $AdS_5 \times S^5/Z_M$ and operators in orbifold field theories*, *JHEP* **09** (2004) 008 [[hep-th/0408014](#)].
- [36] N. Beisert and R. Roiban, *The Bethe ansatz for Z_S orbifolds of $\mathcal{N} = 4$ super Yang-Mills theory*, *JHEP* **11** (2005) 037 [[hep-th/0510209](#)].

- [37] D. Astolfi, V. Forini, G. Grignani and G.W. Semenoff, *Finite size corrections and integrability of $N = 2$ SYM and DLCQ strings on a pp-wave*, *JHEP* **09** (2006) 056 [[hep-th/0606193](#)].
- [38] K.J. Larsen and N.A. Obers, *Phases of thermal $N = 2$ quiver gauge theories*, *JHEP* **01** (2008) 057 [[arXiv:0708.3199](#)].
- [39] D. Astolfi, G. Grignani, T. Harmark and M. Orselli, in preparation.
- [40] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [[hep-th/0109162](#)].
- [41] M. Lubcke and K. Zarembo, *Finite-size corrections to anomalous dimensions in $\mathcal{N} = 4$ SYM theory*, *JHEP* **05** (2004) 049 [[hep-th/0405055](#)].
- [42] T. Bargheer, N. Beisert and N. Gromov, *Quantum stability for the Heisenberg ferromagnet*, [arXiv:0804.0324](#).
- [43] B. Stefański jr. and A.A. Tseytlin, *Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations*, *JHEP* **05** (2004) 042 [[hep-th/0404133](#)].
- [44] S. Bellucci, P.Y. Casteill, J.F. Morales and C. Sochichiu, *SL(2) spin chain and spinning strings on $AdS_5 \times S^5$* , *Nucl. Phys. B* **707** (2005) 303 [[hep-th/0409086](#)].
- [45] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [[hep-th/0507189](#)].
- [46] S. Frolov and A.A. Tseytlin, *Quantizing three-spin string solution in $AdS_5 \times S^5$* , *JHEP* **07** (2003) 016 [[hep-th/0306130](#)].
- [47] S.A. Frolov, I.Y. Park and A.A. Tseytlin, *On one-loop correction to energy of spinning strings in S^5* , *Phys. Rev. D* **71** (2005) 026006 [[hep-th/0408187](#)].
- [48] I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in SL(2) sector*, *JHEP* **03** (2005) 013 [[hep-th/0501203](#)].
- [49] N. Beisert and L. Freyhult, *Fluctuations and energy shifts in the Bethe ansatz*, *Phys. Lett. B* **622** (2005) 343 [[hep-th/0506243](#)].
- [50] G. Grignani, M. Orselli, B. Ramadanovic, G.W. Semenoff and D. Young, *Divergence cancellation and loop corrections in string field theory on a plane wave background*, *JHEP* **12** (2005) 017 [[hep-th/0508126](#)].
- [51] G. Grignani, M. Orselli, B. Ramadanovic, G.W. Semenoff and D. Young, *AdS/CFT vs. string loops*, *JHEP* **06** (2006) 040 [[hep-th/0605080](#)].
- [52] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, [hep-th/9603167](#).
- [53] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett. B* **558** (2003) 229 [[hep-th/0212269](#)].
- [54] N. Beisert, *Higher loops, integrability and the near BMN limit*, *JHEP* **09** (2003) 062 [[hep-th/0308074](#)].
- [55] B. Eden, C. Jarczak and E. Sokatchev, *A three-loop test of the dilatation operator in $\mathcal{N} = 4$ SYM*, *Nucl. Phys. B* **712** (2005) 157 [[hep-th/0409009](#)].
- [56] G. De Risi, G. Grignani, M. Orselli and G.W. Semenoff, *DLCQ string spectrum from $N = 2$ SYM theory*, *JHEP* **11** (2004) 053 [[hep-th/0409315](#)].

- [57] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)].
- [58] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $\mathcal{N} = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [[hep-th/0503192](#)].